

# On sharp rates and analytic compactifications of asymptotically conical Kähler metrics

Chi Li

Let  $X$  be a complex manifold and  $S \hookrightarrow X$  be an embedding of complex submanifold. Assuming that the embedding is  $(k-1)$ -linearizable or  $(k-1)$ -comfortably embedded, we construct via the deformation to the normal cone a diffeomorphism  $F$  from a small neighborhood of the zero section in the normal bundle  $N_S$  to a small neighborhood of  $S$  in  $X$  such that  $F$  is in a precise sense holomorphic to the  $(k-1)$ -th order. Using this  $F$  we obtain optimal estimates on asymptotical rates for asymptotically conical Calabi-Yau metrics constructed by Tian-Yau. Furthermore, when  $S$  is an ample divisor satisfying an appropriate cohomological condition, we relate the order of comfortable embedding to the weight of the deformation of the normal isolated cone singularity arising from the deformation to the normal cone. We also give an example showing that the condition of comfortable embedding depends on the splitting liftings. We then prove an analytic compactification result for the deformation of the complex structure on a complex cone that decays to any positive order at infinity. This can be seen as an analytic counterpart of Pinkham's result on deformations of cone singularities with negative gradings.

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## 1 Introduction

In this paper, by a complex cone  $C(D, L)$ , we will mean the affine variety obtained by contracting the zero section of a negative line bundle  $L^{-1}$  over a smooth projective manifold  $D$ . We will also consider the

compactified cone  $\overline{C}(D, L) = C(D, L) \cup D_\infty$  obtained by adding the divisor  $D_\infty$  at infinity. These varieties can be expressed using pure algebra ( $x$  has degree 1 in the second graded ring):

$$C(D, L) = \text{Spec} \bigoplus_{m=0}^{\infty} H^0(D, mL), \quad \overline{C}(D, L) = \text{Proj} \bigoplus_{m=0}^{\infty} (\oplus_{r=0}^m H^0(D, L^r) \cdot x^{m-r}).$$

Our motivation for this paper is to understand the optimal convergence rate of asymptotically conical (AC) Calabi-Yau Kähler (CY) metrics on non-compact Kähler manifolds. To explain the problem, we consider the AC CY metrics constructed by Tian-Yau. Let  $X$  be a Fano manifold and  $D$  be a smooth divisor such that  $-K_X \sim_{\mathbb{Q}} \alpha D$  with  $\mathbb{Q} \ni \alpha > 1$ . Assume that  $D$  has a Kähler-Einstein metric. Then by Tian-Yau [29], there exists a complete Calabi-Yau metric  $\omega_{\text{TY}}$  on  $M := X \setminus D$  whose metric tangent cone at infinity is some explicit conical Calabi-Yau metric  $\omega_0 = dr^2 + r^2 g_Y$  on the complex cone  $C := C(D, N_D)$  (see Section 3.1). More precisely, there exists a diffeomorphism  $F_K : C(D, N_D) \setminus B_R(\varrho) \rightarrow M \setminus K$  for some compact set  $K \subset\subset M$  such that the following inequality holds:

$$\|\nabla_{\omega_0}^j (F_K^* \omega_{\text{TY}} - \omega_0)\|_{\omega_0} \leq C r^{-\lambda-j} \text{ for } j \geq 0,$$

where  $\nabla_{\omega_0}$  is the Levi-Civita connection of  $\omega_0$ . A natural question is what is the optimal value for the convergence rate  $\lambda_{\max}$ . This is important for gluing constructions of Calabi-Yau metrics on compact manifolds. By the work of Cheeger-Tian [9], Conlon-Hein ([11], [12]), we already have a good understanding of this convergence rate abstractly. Actually, it was shown in [9, Section 7] that for standard Ricci flat Kähler cone, all bounded solutions to linearized Kähler-Ricci-flat equation come either from deformations of Kähler class and decay quadratically, or come from infinitesimal deformations of the complex structure on the base.

More recently, Conlon-Hein [11] studied the solutions to the corresponding complex Monge-Ampère equation for Calabi-Yau metrics to get similar estimates. If we denote by  $\mathfrak{k}$  is Kähler class represented by  $\omega_{\text{TY}}$ , then their estimate of the optimal rate is as follows (see [11], and [13, Remark 1.2]):

$$\lambda_{\max} \geq \begin{cases} \min(2n, \lambda_1), & \text{if } \mathfrak{k} \in H_c^2(M); \\ \min(2, \lambda_1), & \text{if } \mathfrak{k} \in H^2(M). \end{cases} \quad (1)$$

Here  $\lambda_1$  is any number satisfying the following condition: there exists a diffeomorphism  $F_K : C(D, N_D) \setminus B_R(\varrho) \rightarrow M \setminus K$  such that

$$\|\nabla_{\omega_0}^j (F_K^* \Omega - \Omega_0)\|_{\omega_0} \leq C r^{-\lambda_1-j} \text{ for any } j \geq 0, \quad (2)$$

where  $\Omega$  (resp.  $\Omega_0$ ) is the meromorphic volume form on  $X$  (resp.  $\overline{C}(D, N_D)$ ) that is non-vanishing holomorphic on  $M = X \setminus D$  (resp.  $C(D, N_D)$ ) and has pole of order  $\alpha$  along  $D$ . Conlon-Hein [11] showed that the condition (2) implies the following condition:

$$\|\nabla_{\omega_0}^j (F_K^* J - J_0)\|_{\omega_0} \leq C r^{-\lambda_1-j} \text{ for any } j \geq 0, \quad (3)$$

where  $J$  (resp.  $J_0$ ) is the complex structure on  $M$  (resp.  $C(D, N_D)$ ). So we see that  $\lambda_1$  essentially measures the difference between the complex structure of  $M \setminus K$  and  $C(D, N_D) \setminus B_R(\varrho)$ . It's easy to see that, equivalently we are comparing the complex structure on the (punctured) neighborhood of  $D$  inside  $X$  and the complex structure of (punctured) neighborhood of  $D$  inside  $N_D$ .

In general, let  $S$  be a complex submanifold of an ambient complex manifold  $X$ . The comparison between neighborhoods of  $S$  inside  $X$  with neighborhoods of  $S$  inside the normal bundle  $N_S$  was studied in depth in depth by Grauert ([16]), Griffiths ([17]), and Camacho-Movasati-Sad ([7], [8]). Note that although that in general  $N_S$  has a different holomorphic structure with that of neighborhood of  $S$  inside  $X$ ,  $N_S$  can be viewed as a first order approximation of small neighborhood of  $S$ . Denote by  $S(k)$  the ringed analytic space  $(S, \mathcal{O}_X/\mathcal{I}_S^{k+1})$ , which will be called the  $k$ -th infinitesimal neighborhood of  $S$  inside  $X$ .

**Definition 1.1.**  *$S$  is  $k$ -linearizable inside  $X$  if its  $k$ -th infinitesimal neighbourhood  $S(k)$  in  $X$  is isomorphic to its  $k$ -th infinitesimal neighbourhood  $S_N(k)$  in  $N_S$ . Here we identify  $S$  with the zero section  $S_0$  of  $N_S =: N$ .*

Grauert [16] showed that the obstruction for extending an isomorphism  $S(k-1) \rightarrow S_N(k-1)$  to an isomorphism  $S(k) \rightarrow S_N(k)$  lies in the cohomology group  $H^1(S, \Theta_X|_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1})$ . He also pointed out that this obstruction consists of two parts. To see this, consider the exact sequence:

$$0 \rightarrow \Theta_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1} \rightarrow \Theta_X|_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1} \rightarrow N_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1} \rightarrow 0,$$

from which we get the long exact sequence:

$$\cdots \rightarrow H^1(S, \Theta_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1}) \rightarrow H^1(S, \Theta_X|_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1}) \rightarrow H^1(S, N_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1}) \rightarrow \cdots$$

So roughly speaking, the obstruction comes from two parts, one from  $H^1(S, N_S \otimes \mathcal{I}_S^k / \mathcal{I}_S^{k+1})$  and the other from  $H^1(S, \Theta_S \otimes \mathcal{I}_S^k / \mathcal{I}_S^{k+1})$ . In [1], Abate-Bracci-Tovena explicitly described these two cohomological obstruction classes, and introduced the notion of  $k$ -splitting and  $k$ -comfortably embedded such that  $k$ -linearizable =  $k$ -splitting +  $(k-1)$ -comfortably embedded with respect to the induced  $(k-1)$ -th order lifting. For more details, see Appendix 5.2.

Returning back to our goal of estimating optimal rate, to get condition (2) and (3) we want to compare the two embeddings  $S \subset X$  and  $S_0 \subset N_S$  by constructing a diffeomorphism which is in some sense the most holomorphic one. This might be known to experts after the celebrated work of Grauert [16]. For example, some rough arguments appeared in [31] (see also [3]). Here we would like to give an almost explicit construction using the work of Abate-Bracci-Tovena [1] together with the deformation to the normal cone construction. We first state a preliminary result. Let  $\tilde{g}_0$  be a smooth Riemannian metric on a neighborhood  $W_0$  of  $S_0$  inside  $N_S$ . Denote by  $\|\cdot\|_{\tilde{g}_0}$  the  $C^0$ -norms of tensors on  $W_0$  with respect to  $\tilde{g}_0$  and by  $\tilde{r}$  the distance function to  $S_0$  with respect to  $\tilde{g}_0$ .

**Proposition 1.1.** *Assume  $S \hookrightarrow X$  is  $(k-1)$ -linearizable, then there exists a diffeomorphism  $F : W_0 \rightarrow F(W_0) \subset W$  where  $W$  is a small neighborhood of  $S \subset X$ , such that  $F$  satisfies*

$$\|\nabla_{\tilde{g}_0}^j (F^*J - J_0)\|_{\tilde{g}_0} \leq \tilde{r}^{k-j}, \text{ for any } j \geq 0. \quad (4)$$

Here we point out that the norm used in (4) is with respect to  $\tilde{g}_0$  while the norms used in (2)-(3) are with respect to the cone metric  $\omega_0$  (see Section 3.1 for the comparison between these two Kähler metrics). This discrepancy is explained by the difference between linearizable and comfortable embedding. In other words, if we assume that  $S$  is an ample divisor  $D$  as that appeared at the beginning of the introduction and  $D$  is furthermore  $(k-1)$ -comfortably embedded, we get exactly the conditions (2)-(3) that we want.

**Proposition 1.2.** *Assume that  $D$  is  $(k-1)$ -comfortably embedded. Then there exists a diffeomorphism away from compact sets  $F_K : C(D, N_D) \setminus B_R(\partial) \rightarrow (X \setminus D) \setminus K$  such that*

$$\|\nabla_{\omega_0}^j (F_K^* \Omega - \Omega_0)\|_{\omega_0} \leq r^{-\frac{nk}{\alpha-1}-j}, \text{ and } \|\nabla_{\omega_0}^j (F_K^*J - J_0)\|_{\omega_0} \leq r^{-\frac{nk}{\alpha-1}-j}. \quad (5)$$

Here the number  $\frac{\alpha-1}{n}$  is the inverse of the exponent in the Calabi-ansatz (see (36) in Section 3.1). We will see in Theorem 1.2 that the estimates in (5) should be sharp. As a consequence, we immediately get a lower bound estimate of the optimal rate of some class of AC Calabi-Yau metrics constructed by Tian-Yau [29].

**Corollary 1.1.** *Using the above notation, the Tian-Yau metric  $\omega_{TY}$  satisfies:*

$$\|\nabla_{\omega_0}^j (F_K^* \omega_{TY} - \omega_0)\|_{\omega_0} \leq r^{-\min\{2, \frac{nk}{\alpha-1}\}-j} \text{ for } j \geq 0.$$

If moreover we assume that the Kähler class is contained in the compactly supported cohomology  $H_c^2(X \setminus D)$ , then we get:

$$\|\nabla_{\omega_0}^j (F_K^* \omega_{TY} - \omega_0)\|_{\omega_0} \leq r^{-\min\{2n, \frac{nk}{\alpha-1}\}-j} \text{ for } j \geq 0$$

As discussed before, the corollary follows from Proposition 1.2 and the regularity theory developed by Conlon-Hein [11]. The last statement also follows implicitly from Proposition 1.2 and Cheeger-Tian's work [9]. In many cases, Proposition 1.2 improves the regularity in [12] (see also [13, Remark 1.2]). We refer to Appendix 5.1, in particular, Remark 5.2, for more background details.

As mentioned above, the construction of diffeomorphism  $F$  in above Propositions uses a standard construction in algebraic geometry which is called deformation to the normal cone. This is a way to degenerate a neighborhood of  $S \hookrightarrow X$  to a neighborhood of  $S \hookrightarrow N_S$ . The construction is simply to blown-up the submanifold  $S \times \{0\} \subset X \times \mathbb{C}$  which gives a total family  $\tilde{\mathcal{X}} = Bl_{S \times \{0\}}(X \times \mathbb{C})$  with the projection  $\pi : \tilde{\mathcal{X}} \rightarrow \mathbb{C}$ . The central fibre  $\tilde{\mathcal{X}}_0 = X \cup E$  is the union of two components. The exceptional divisor  $E = \mathbb{P}(N_S \oplus \mathbb{C})$  is the projective compactification of the normal bundle  $N_S$  of  $S \subset X$ . In this way we can view  $S \hookrightarrow X$  as an analytic deformation of  $S_0 \hookrightarrow N_S$  and try to read out the order of difference from deformation theory.

To move further in this direction, we assume  $S = D$  is an ample divisor in  $X$  from now on. Our next result relates the number  $k$  in the above proposition to the weight of deformation of some singularity. The component  $X \subset \tilde{\mathcal{X}}_0$  from above can be contracted to a point. See figure 1 for illustration. In this way, we get a flat family of irreducible projective varieties  $\mathcal{X} \rightarrow \mathbb{C}$ . For simplicity, we will say that  $\mathcal{X}$  is the degeneration obtained from the *Contracted Deformation to the Normal Cone associated to  $(X, D)$* . There are equivalent ways to realize this construction, one using the graph construction (see section 2.2.2), and another using pure algebra (see Proof of Lemma 2.2).

Under appropriate cohomological assumptions (see Section 2.2.2), the contracted central fibre  $\mathcal{X}_0$  is normal and hence coincides with the projective cone  $\bar{C} = \bar{C}(D, N_D)$ . Then we get a  $\mathbb{C}^*$ -equivariant degeneration  $\mathcal{X}$  of  $X$  to the projective cone  $\bar{C} = \bar{C}(D, N_D)$  which in general has an isolated singularity  $\underline{o}$ . Note that  $\underline{o}$  is simply the image of the infinity divisor  $D_\infty$  of  $E = \mathbb{P}(N_D \oplus \mathbb{C})$  under the contraction. Denote by  $\mathcal{D} \cong D \times \mathbb{C}$  the strict transform of  $D \times \mathbb{C}$  in this process. Then the variety  $\mathcal{X}^\circ = \mathcal{X} \setminus \mathcal{D}$  is a degeneration of quasi-

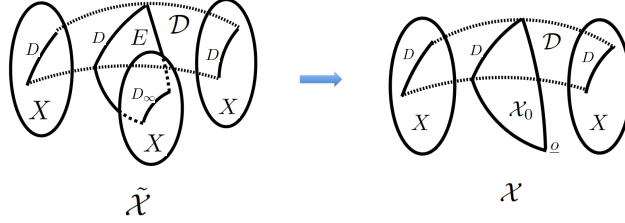


Figure 1:  $\mu : \tilde{\mathcal{X}} \longrightarrow \mathcal{X}$

projective variety  $X \setminus D$  to the affine cone  $C = C(D, N_D) = \bar{C} \setminus D$ . In other words,  $\mathcal{X}^\circ$  can be viewed as a deformation of the cone  $C$ . Let  $\mathcal{C} \rightarrow \text{Def}(C)$  be the versal deformation of  $C$ . Then  $\mathcal{X}^\circ$  is induced from a map  $\mathbf{I}_{\mathcal{X}^\circ} : \mathbb{C} \rightarrow \text{Def}(C)$ . By Kuranishi-Grauert,  $\text{Def}(C)$  is a locally complete analytic variety in  $\mathbf{T}_C^1$ . So  $\mathbf{I}_{\mathcal{X}^\circ}$  is the germ of a vector-valued holomorphic function whose image passes through  $\mathbf{0} \in \mathbf{T}_C^1$  and lies in  $\text{Def}(C) \subset \mathbf{T}_C^1$ . Denote  $\kappa = \text{ord}_0 \mathbf{I}_{\mathcal{X}^\circ}$  the vanishing order of the  $\mathbf{I}_{\mathcal{X}^\circ}$  at  $0 \in \mathbb{C}$ . We define the reduced Kodaira-Spencer class to be

$$\mathbf{KS}_{\mathcal{X}^\circ}^{\text{red}} = \frac{1}{\kappa!} \left. \frac{d^\kappa}{dt^\kappa} \right|_{t=0} \mathbf{I}_{\mathcal{X}^\circ}(t). \quad (6)$$

By the works of Schlessinger ([27]), Pinkham ([25], [26]) and Wahl ([30]), we have a good understanding of  $\mathbf{T}_C^1$  which is a graded vector space. We denote  $w(X, D)$  the weight of  $\mathbf{KS}_{\mathcal{X}^\circ}^{\text{red}}$ . Because  $\mathcal{X}^\circ$  comes from  $\mathcal{X}$  that is also a deformation of  $\bar{C}(D, N_D) \setminus \{\underline{o}\} = (N_D \rightarrow D)$ , it's easy to get that  $w(X, D) \leq 0$ . On the other hand, we define the integer  $m(X, D)$  to be the maximum positive integer  $m$  such that the embedding  $D \hookrightarrow X$  is  $(m-1)$ -comfortably embedded (see [1, Remark 4.6]). We then have:

**Theorem 1.1.** *In the above setup, i.e. we assume the contracted deformation normal cone associated with  $(X, D)$  degenerates  $X$  to  $\bar{C}(X, N_D)$ . Assume furthermore that  $n \geq 3$ . Then  $|w(X, D)| = m(X, D)$ .*

If  $\dim D \geq 2$ , by remark 5.4,  $m(X, D)$  is also the maximal order of linearizable embedding. In other words,  $D \subset X$  is  $(m(X, D) - 1)$ -linearizable but not  $m(X, D)$ -linearizable. When  $\dim D = 1$ , we expect the conclusion of Theorem 1.1 is also true. In fact, a parallel analytic result will be shown in Theorem 1.2 without the restriction on dimension. On the other hand, we will calculate the example of diagonal embedding  $\mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  explicitly to see some related new phenomenon about the embedding of submanifolds in Proposition 2.3. In particular, this example shows that the condition of comfortable embedding depends on the choice of splitting liftings, and thus answers a question by Abate-Bracci-Tovena negatively.

Combining this result with Proposition 1.2, we can give simple algebraic interpretations of ad hoc calculations in [11] on the asymptotical rate of holomorphic volume forms. See Examples in Section 3.2.

Reversely, it's natural to ask if any deformation of complex structure on  $C$  that decays at infinity comes from this construction. We have a good understanding of the algebraic version of this problem thanks to the work of Pinkham. His results in particular implies that any (formal) deformation of  $C$  with negative weight can be extended to a (formal) deformation of  $\bar{C}$  (see Theorem 5.8). For our application to the study of asymptotical conical Kähler metrics, we prove an analytic compactification result, which can be seen the analytic counterpart of Pinkham's result. Note that a similar compactification result in the asymptotically cylindrical Calabi-Yau case has recently appeared in [18]. See Remark 4.2 for some comparison. To state this result in a slightly more general form, let  $h$  be a Hermitian metric on any negative line bundle  $L^{-1} \rightarrow D$  with negative Chern curvature. Since  $C = C(D, L)$  is obtained from  $L^{-1}$  by contracting the zero section,  $h$  can be thought as a function on the cone  $C$ . Fixing any  $\delta > 0$ , there is a complete Kähler cone metrics on  $C(D, L)$  given as (see section 3.1)

$$g_0 = \sqrt{-1} \partial \bar{\partial} h^\delta = dr^2 + r^2 g_Y.$$

Let  $U_\epsilon$  be a neighborhood of the infinity end of  $C(D, L)$ . Equivalently  $U_\epsilon$  is a punctured neighborhood of the embedding  $D \hookrightarrow \bar{C}(D, L)$ . Denote  $J_0$  the standard complex structure on  $C(D, L)$ , and  $\bar{U}_\epsilon = U_\epsilon \cup D$  the compactification of  $U_\epsilon$  in  $\bar{C}(D, L)$ .

**Theorem 1.2.** Assume that  $J$  is a complex structure on  $U_\epsilon = \overline{U}_\epsilon \setminus D$  such that

$$\|\nabla_{g_0}^k (J - J_0)\|_{g_0} \leq r^{-\lambda-k},$$

for some  $\lambda > 0$ . Then the complex analytic structure on  $U_\epsilon$  extends to a complex analytic structure on  $\overline{U}_\epsilon$ . Moreover, if we denote by  $m = \lceil \delta\lambda \rceil$  the minimal integer which is bigger than or equal to  $\delta\lambda$ , then in the compactification the divisor  $D$  is  $(m-1)$ -comfortably embedded.

This can be seen as a converse to the Proposition 1.2 and implies the estimate in Proposition 1.2 is sharp.

**Remark 1.1.** Because our proof uses only locally information near the divisor, the argument in the proof should apply in the more general orbifold case. Actually Conlon-Hein [13] has recently used the compactification obtained in Theorem 1.2 to prove any AC CY metric with quasi-regular tangent cone at infinity comes from Tian-Yau's construction.

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## 2 Embeddings of submanifolds and deformations

### 2.1 Construction of comparison diffeomorphism

Under appropriate conditions on the embedding of  $S \subset X$ , we will construct a diffeomorphism which is holomorphic up to some precise order. The idea is to first use the deformation to the normal cone to construct a (local) differentiable family. Then we use the method as in Kodaira's book [20, Section 2.3] to construct the diffeomorphism. The work of Abate-Bracci-Tovena [1] allows us to read out the order of holomorphicity. We refer to Appendix 5.2 for the definitions and preliminary results from [1] we need for the following discussions. We consider two slightly different conditions.

1.  $((k-1)$ -linearizable) By Theorem 5.6 in Appendix 5.2 we can find coordinate charts  $\{V_\alpha, \{z_\alpha\}\}$  of  $X$  near the submanifold  $S$  such that  $S \cap V_\alpha = \{z_\alpha^1 = \dots = z_\alpha^m = 0\}$  and the transition functions on  $V_\alpha \cap V_\beta$  are given by:

$$\begin{cases} z_\beta^r &= \sum_{s=1}^m (a_{\beta\alpha})_s^r (z_\alpha^s) z_\alpha^s + R_k^r, & \text{for } r = 1, \dots, m, \\ z_\beta^p &= \phi_{\beta\alpha}^p(z_\alpha'') + R_k^p, & \text{for } p = m+1, \dots, n. \end{cases} \quad (7)$$

Here  $R_k^r, R_k^p \in \mathcal{I}_S^k$ . We also consider coordinate charts  $\{V_\alpha \times \mathbb{C}, \{z_\alpha, t\}\}$  on  $X \times \mathbb{C}$  so that  $S \times \{0\} = \{z_\alpha^1 = \dots = z_\alpha^m = t = 0\}$ .

Consider the blow up  $\pi: \tilde{\mathcal{X}} := Bl_{S \times \{0\}}(X \times \mathbb{C}) \rightarrow X \times \mathbb{C}$  with the exceptional divisor  $E = \mathbb{P}(N_S \oplus \mathbb{C})$ .  $E$  is the projective compactification of the normal bundle  $N_S \rightarrow S$  and  $S_0$  sits inside  $N_S \subset E \subset \tilde{\mathcal{X}}_0 \subset \tilde{\mathcal{X}}$  as the zero section of  $N_S \rightarrow S$ . The subset  $\pi^{-1}(V_\alpha \times \mathbb{C}) \subset \tilde{\mathcal{X}}$  is defined as a subvariety of  $V_\alpha \times \mathbb{C} \times \mathbb{P}^m$ :

$$\begin{aligned} \{(z_\alpha^r, z_\alpha^p, t, [Z_\alpha^r, T]); (z_\alpha^r, z_\alpha^p) \in V_\alpha, t \in \mathbb{C}, z_\alpha^r Z_\alpha^s - z_\alpha^s Z_\alpha^r = 0, \\ z_\alpha^r \cdot T - t \cdot Z_\alpha^r = 0; \text{ for } r, s = 1, \dots, m; p = m+1, \dots, n\}. \end{aligned}$$

where  $[Z_\alpha^r, T]$  are homogenous coordinates on  $\mathbb{P}^m$ . Near  $S_0$ , the coordinate  $T \neq 0$ , and so we can define new coordinate charts  $\{w_\alpha, t\}$  such that the map  $\pi$  is given by:

$$z_\alpha^1 = tw_\alpha^1, \dots, z_\alpha^m = tw_\alpha^m; \quad z_\alpha^{m+1} = w_\alpha^{m+1}, \dots, z_\alpha^n = w_\alpha^n; \quad t = t.$$

Without loss of generality we can assume  $V_\alpha = \{z_\alpha; |z_\alpha| < \epsilon\}$  for sufficiently small  $\epsilon > 0$ . Then if we denote the polydisc on the total space:

$$\mathcal{U}_\alpha = \{(t, w_\alpha); |t| < 1, |w_\alpha| < \epsilon\},$$

then  $\pi(\mathcal{U}_\alpha) \subset V_\alpha \times \mathbb{C}$ , and when  $t \neq 0$ ,

$$\pi(\mathcal{U}_\alpha) \cap X_t \cong \{z_\alpha; |z_\alpha^r| < \epsilon t, |z_\alpha^p| < \epsilon; \text{ for } r = 1, \dots, m; p = m+1, \dots, n\}.$$

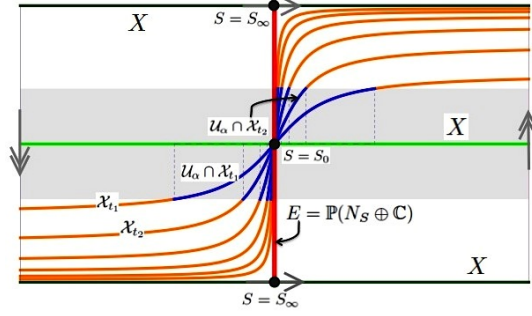


Figure 2: Deformation to the normal cone: graph construction

See figure 2 for a schematic illustration of deformation to the normal cone using the graph construction. See also [14, Remark 5.1.1, Section 5.1] and Section 2.2.2. Denote by  $\mathcal{S}$  the strict transform of  $S \times \mathbb{C}$  on  $\tilde{\mathcal{X}}$ . Then the collection of open sets  $\{\mathcal{U}_\alpha\}$  is a covering of  $\mathcal{S}$  inside the total space  $\tilde{\mathcal{X}}$  and on  $\mathcal{U}_\alpha$  the ideal sheaf  $\mathcal{I}_\mathcal{S}$  is generated by  $w_\alpha^1, \dots, w_\alpha^m$ . Denote  $\mathcal{U} = \bigcup_\alpha \mathcal{U}_\alpha$ . We can find a small neighborhood  $\mathcal{W}$  of  $\mathcal{S} \subset \tilde{\mathcal{X}}$  such that  $\mathcal{W} \subset \mathcal{U}$ . Denote  $w'_\alpha = (w_\alpha^1, \dots, w_\alpha^m)$ ,  $w''_\alpha = (w_\alpha^{m+1}, \dots, w_\alpha^n)$  and define

$$\tilde{R}_k^r(t; w'_\alpha, w''_\alpha) = t^{-k} R_k^r(t w'_\alpha, w''_\alpha), \quad \tilde{R}_k^p(t; w'_\alpha, w''_\alpha) = t^{-k} R_k^p(t w'_\alpha, w''_\alpha).$$

Then  $\tilde{R}_k^r \in \mathcal{I}_\mathcal{S}^k$ ,  $\tilde{R}_k^p \in \mathcal{I}_\mathcal{S}^k$ . Note that  $\{\mathcal{U}_\alpha \cap \mathcal{W}_t, w_\alpha\}_\alpha$  form coordinate chart covering of  $\mathcal{W}_t := \pi^{-1}(X_t) \cap \mathcal{W}$ . The transition function on  $(\mathcal{U}_\alpha \cap \mathcal{W}_t) \cap (\mathcal{U}_\beta \cap \mathcal{W}_t)$  is given by:

$$\begin{cases} w_\beta^r &= \sum_{s=1}^m (a_{\beta\alpha})_s^r (w''_\alpha)^s + t^{k-1} \tilde{R}_k^r, & \text{for } r = 1, \dots, m, \\ w_\beta^p &= \phi_{\beta\alpha}^p(w''_\alpha) + t^k \tilde{R}_k^p, & \text{for } p = m+1, \dots, n. \end{cases} \quad (8)$$

Now we choose a partition of unity  $\{\rho_\alpha, \bar{\rho}\}$  subordinate to the covering  $\{\mathcal{U}_\alpha, \tilde{\mathcal{X}} \setminus \overline{\mathcal{W}}\}$ . In particular,  $\text{Supp}(\rho_\alpha) \subset \mathcal{U}_\alpha$ ,  $\text{Supp}(\bar{\rho}) \cap \mathcal{W} = \emptyset$ . As in Appendix 5.3, define the differentiable vector field in the small neighborhood  $\mathcal{W}$  of  $\mathcal{S} \subset \tilde{\mathcal{X}}$ :

$$\begin{aligned} \mathbb{V} &= \sum_\alpha \rho_\alpha \left( \frac{\partial}{\partial t} \right)_\alpha = \sum_{i=1}^n \left( \sum_\alpha \rho_\alpha \frac{\partial f_{\beta\alpha}^i(w_\alpha, t)}{\partial t} \right) \frac{\partial}{\partial w_\beta^i} + \left( \frac{\partial}{\partial t} \right)_\beta \\ &= \sum_{r=1}^m \sum_\alpha \rho_\alpha \partial_t (t^{k-1} \tilde{R}_k^r) \frac{\partial}{\partial w_\beta^r} + \sum_{p=m+1}^n \sum_\alpha \rho_\alpha \partial_t (t^k \tilde{R}_k^p) \frac{\partial}{\partial w_\beta^p} + \left( \frac{\partial}{\partial t} \right)_\beta. \end{aligned}$$

Let  $\sigma(t)$  be the flow generated by  $\mathbb{V}$  which exists when  $|t| \leq \delta$  for sufficiently small  $\delta$ . Note that the vector field  $V$  is tangent to  $\mathcal{S}$  so that  $\sigma(t)$  preserves  $\mathcal{S}$ . Denote  $\mathcal{J}$  the complex structure on the total space  $\tilde{\mathcal{X}}$  of blow up. Denote

$$\Phi(t) = \sigma(t)^* \mathcal{J} - \mathcal{J}.$$

Then we can calculate:

$$\begin{aligned} \dot{\Phi}(t) &= \frac{d}{dt} (\sigma(t)^* \mathcal{J}) = \mathcal{L}_\mathbb{V} \mathcal{J} = \bar{\partial} \mathbb{V} \\ &= \sum_{r=1}^m \sum_\alpha [\partial_t (t^{k-1} \tilde{R}_k^r)] (\bar{\partial} \rho_\alpha) \otimes \frac{\partial}{\partial w_\beta^r} + \sum_{p=m+1}^n \sum_\alpha [\partial_t (t^k \tilde{R}_k^p)] (\bar{\partial} \rho_\alpha) \otimes \frac{\partial}{\partial w_\beta^p} \end{aligned}$$

Assume  $\tilde{\omega}_0$  is a smooth Kähler metric on the open set  $\mathcal{W}$ . Because both  $\tilde{R}_k^r, \tilde{R}_k^p \in \mathcal{I}_\mathcal{S}^k$ , we get:

$$|\dot{\Phi}|_{\tilde{\omega}_0} \leq C t^{\max\{0, k-2\}} |w'|^k.$$

So we can integrate to get:

$$|\Phi(t)|_{\tilde{\omega}_0} = |\sigma(t)^* \mathcal{J} - \mathcal{J}|_{\tilde{\omega}_0} \leq C t^{k-1} |w'|^k. \quad (9)$$

When  $0 < |t| < t_1$  for  $t_1$  sufficiently small, we get a map  $\sigma(t) : \mathcal{W} \cap \tilde{\mathcal{X}}_0 \rightarrow \mathcal{U} \cap \tilde{\mathcal{X}}_t$  which gives a diffeomorphism to its image. By construction,  $\mathcal{W} \cap \tilde{\mathcal{X}}_0$  is a small neighborhood of  $S_0$  and  $\mathcal{U} \cap \tilde{\mathcal{X}}_t$  is a small neighborhood of  $S \subset X = X_t$ .

2.  $((k-1)$ -comfortably embedded) In this case, we can improve the order of some components. This will also be reflected in later discussions. By Theorem 5.5, we can choose the coordinate charts such that the following holds:

$$\begin{cases} z_\beta^r &= \sum_{s=1}^m (a_{\beta\alpha})_s^r (z_\alpha'') z_\alpha^s + R_{k+1}^r, & \text{for } r = 1, \dots, m, \\ z_\beta^p &= \phi_{\beta\alpha}^p(z_\alpha'') + R_k^p, & \text{for } p = m+1, \dots, n. \end{cases} \quad (10)$$

where  $R_{k+1}^r \in \mathcal{I}_S^{k+1}$ ,  $R_k^p \in \mathcal{I}_S^k$ . Similarly as before, denote  $\tilde{R}_{k+1}^r(t; w'_\alpha, w''_\alpha) = t^{-(k+1)} R_{k+1}^r(t w'_\alpha, w''_\alpha)$  and  $\tilde{R}_k^p(t; w'_\alpha, w''_\alpha) = t^{-k} R_k^p(t w'_\alpha, w''_\alpha)$ . Then  $\tilde{R}_{k+1}^r \in \mathcal{I}_S^{k+1}$  and  $\tilde{R}_k^p \in \mathcal{I}_S^k$ . On the total space of the deformation to the normal cone, we have

$$\begin{cases} w_\beta^r &= \sum_{s=1}^m (a_{\beta\alpha})_s^r (w''_\alpha) w_\alpha^s + t^k \tilde{R}_{k+1}^r, & \text{for } r = 1, \dots, m, \\ w_\beta^p &= \phi_{\beta\alpha}^p(w''_\alpha) + t^k \tilde{R}_k^p, & \text{for } p = m+1, \dots, n. \end{cases} \quad (11)$$

Similarly as before the differentiable vector field  $\mathbb{V}$  (see Appendix 5.3) becomes

$$\begin{aligned} \mathbb{V} &= \sum_{i=1}^n \left( \sum_{\alpha} \rho_\alpha \frac{\partial f_{\beta\alpha}^i(w_\alpha, t)}{\partial t} \right) \frac{\partial}{\partial w_\beta^i} + \left( \frac{\partial}{\partial t} \right)_\beta \\ &= \sum_{r=1}^m \sum_{\alpha} \rho_\alpha [\partial_t(t^k \tilde{R}_{k+1}^r)] \otimes \frac{\partial}{\partial w_\beta^r} + \sum_{p=m+1}^n \sum_{\alpha} \rho_\alpha [\partial_t(t^k \tilde{R}_k^p)] \otimes \frac{\partial}{\partial w_\beta^p} + \left( \frac{\partial}{\partial t} \right)_\beta. \end{aligned} \quad (12)$$

Use the same notations  $\sigma(t)$ ,  $\mathcal{J}$ ,  $\Phi(t)$  and  $\dot{\Phi}(t)$  as before. We have:

$$\begin{aligned} \dot{\Phi}(t) &= \frac{d}{dt}(\sigma(t)^* \mathcal{J}) = \mathcal{L}_\mathbb{V} \mathcal{J} = \bar{\partial} \mathbb{V} \\ &= \sum_{r=1}^m \sum_{\alpha} [\partial_t(t^k \tilde{R}_{k+1}^r)] (\bar{\partial} \rho_\alpha) \otimes \frac{\partial}{\partial w_\beta^r} + \sum_{p=m+1}^n \sum_{\alpha} [\partial_t(t^k \tilde{R}_k^p)] (\bar{\partial} \rho_\alpha) \otimes \frac{\partial}{\partial w_\beta^p}. \end{aligned}$$

We assume the index  $v \in \{1, \dots, m, \bar{1}, \dots, \bar{m}\}$ ,  $h \in \{m+1, \dots, n, \overline{m+1}, \dots, \bar{n}\}$  and decompose  $\Phi$  into four types of components:

$$\Phi = \Phi_v^h + \Phi_h^v + \Phi_v^v + \Phi_h^h := \phi_v^h dw^v \otimes \partial_{w^h} + \phi_h^v dw^h \otimes \partial_{w^v} + \phi_v^v dw^v \otimes \partial_{w^v} + \phi_h^h dw^h \otimes \partial_{w^h}.$$

Again we assume  $\tilde{\omega}_0$  is a smooth Kähler metric on  $\mathcal{W}$ . Since  $\tilde{R}_{k+1}^r \in \mathcal{I}_S^{k+1}$ ,  $\tilde{R}_k^p \in \mathcal{I}_S^k$ , it's easy to see that:

$$|\dot{\phi}_h^v| \leq Ct^{k-1} |w'|^{k+1}, |\dot{\phi}_v^v| \leq Ct^{k-1} |w'|^{k+1}, |\dot{\phi}_h^h| \leq Ct^{k-1} |w'|^k, |\dot{\phi}_v^h| \leq Ct^{k-1} |w'|^k.$$

Integrating these, we get:

$$|\Phi_h^v|_{\tilde{\omega}_0} \leq Ct^k |w'|^{k+1}, |\Phi_v^v|_{\tilde{\omega}_0} \leq Ct^k |w'|^{k+1}, |\Phi_h^h|_{\tilde{\omega}_0} \leq Ct^k |w'|^k, |\Phi_v^h|_{\tilde{\omega}_0} \leq Ct^k |w'|^k.$$

Again when  $|t|$  is sufficiently small, we get the estimates, which improve the estimates in (9) for the horizontal-to-vertical and vertical-to-vertical terms.

## 2.2 Deformation weight and order of comfortable embedding

### 2.2.1 Order of embedding via deformation to the normal cone

Let  $S$  be a smooth submanifold of a complex manifold  $X$ . We will denote by  $\pi_S : N_S \rightarrow S$  the normal bundle of  $S$  inside  $X$  and by  $\Theta_{N_S}$  the tangent sheaf on the total space  $N_S$ . The natural  $\mathbb{C}^*$  action on  $N_S$  induces  $\mathbb{C}^*$  actions on various cohomology groups. Since we will use various Čech cohomology groups frequently, we choose a Stein covering  $\{\hat{U}_\alpha\}$  of  $N_S$  by first choosing a Stein covering  $\{U_\alpha\}$  of  $S$  and then defining  $\hat{U}_S = \pi_S^{-1}(U_\alpha)$ . In particular,  $\hat{U}_\alpha$  is invariant under the natural  $\mathbb{C}^*$  action. On each  $\hat{U}_\alpha$ , we choose



a coordinate system  $w_\alpha = \{w'_\alpha, w''_\alpha\} = \{w^r_\alpha, w^p_\alpha; r = 1, \dots, m; p = m+1, \dots, n\}$  such that  $w^r_\alpha$  are fiber variables and  $w^p_\alpha$  are base variables. In particular, the transition function on  $\hat{U}_\alpha \cap \hat{U}_\beta$  is of the form:

$$\begin{cases} w^r_\beta &= \sum_{s=1}^m (a_{\beta\alpha})^r_s (w''_\alpha) w^s_\alpha, & \text{for } r = 1, \dots, m, \\ w^p_\beta &= \phi^p_{\beta\alpha}(w''_\alpha), & \text{for } p = m+1, \dots, n. \end{cases} \quad (13)$$

In the following, if  $\mathbb{C}^*$  acts on a vector space  $\mathbf{V}$ , then we will denote by  $\mathbf{V}(-k)$  the  $(-k)$ -weight space in the weight decomposition of  $\mathbf{V}$ .

**Lemma 2.1.** *For  $k \geq 0$ , we have the following commutative diagram of exact sequences*

$$\begin{array}{ccccc} H^1(N_S, \Theta_{N_S} \otimes \mathcal{I}_S^{k+1})(-k) & \xrightarrow{\mathfrak{N}'_k} & H^1(N_S, \Theta_{N_S} \otimes \mathcal{I}_S^k)(-k) & \xrightarrow{\mathfrak{T}'_k} & H^1(S, \Theta_S \otimes \mathcal{I}_S^k / \mathcal{I}_S^{k+1}) \\ \mathfrak{R}_k \downarrow \cong & & \mathfrak{J}_k \downarrow \cong & & \parallel \\ H^1(S, N_S \otimes \mathcal{I}_S^{k+1} / \mathcal{I}_S^{k+2}) & \xrightarrow{\mathfrak{N}_k} & H^1(N_S, \Theta_{N_S})(-k) & \xrightarrow{\mathfrak{T}_k} & H^1(S, \Theta_S \otimes \mathcal{I}_S^k / \mathcal{I}_S^{k+1}) \end{array} \quad (14)$$

*Proof.* We first notice that  $\mathfrak{T}'_k$  is well defined as the composition of maps:

$$H^1(N_S, \Theta_{N_S} \otimes \mathcal{I}_S^k) \rightarrow H^1(S, \Theta_{N_S}|_S \otimes \mathcal{I}_S^k / \mathcal{I}_S^{k+1}) \rightarrow H^1(S, \Theta_S \otimes \mathcal{I}_S^k / \mathcal{I}_S^{k+1}).$$

In the last map, we used the holomorphic splitting  $\Theta_{N_S}|_S = \Theta_S \oplus N_S$ . Similarly  $\mathfrak{R}_k$  is well defined as the composition of maps:

$$H^1(N_S, \Theta_{N_S} \otimes \mathcal{I}_S^{k+1}) \rightarrow H^1(S, \Theta_{N_S}|_S \otimes \mathcal{I}_S^{k+1} / \mathcal{I}_S^{k+2}) \rightarrow H^1(S, N_S \otimes \mathcal{I}_S^{k+1} / \mathcal{I}_S^{k+2}).$$

Let's first show that the first row of sequence is exact. Let  $\theta_k \in H^1(N_S, \Theta_{N_S} \otimes \mathcal{I}_S^k)(-k)$  be represented by a weight  $(-k)$  cocycle:

$$(\theta_k)_{\beta\alpha} = \sum_{r=1}^m b^r_{\beta\alpha}(w) \frac{\partial}{\partial w^r_\beta} + \sum_{p=m+1}^n c^p_{\beta\alpha}(w) \frac{\partial}{\partial w^p_\beta},$$

where  $b^r_{\beta\alpha}, c^p_{\beta\alpha} \in \mathcal{I}_S^k$ . Since  $\frac{\partial}{\partial w^r_\beta}$  (resp.  $\frac{\partial}{\partial w^p_\beta}$ ) has weight 1 (resp. 0), we know that  $b^r_{\beta\alpha}$  (resp.  $c^p_{\beta\alpha}$ ) is homogeneous of degree  $(k+1)$  (resp.  $k$ ) in  $w' = \{w^r_\beta\}$ . Then

$$(\mathfrak{T}'_k(\theta_k))_{\beta\alpha} = \sum_{p=m+1}^n [c^p_{\beta\alpha}(w)]_{k+1} \frac{\partial}{\partial w^p_\beta}.$$

If  $\mathfrak{T}'_k(\theta_k) = 0$ , then we can write:

$$[c^p_{\beta\alpha}(w)]_{k+1} \frac{\partial}{\partial w^p_\beta} = [d^p_\beta]_{k+1} \frac{\partial}{\partial w^p_\beta} - [d^q_\alpha]_{k+1} \frac{\partial}{\partial w^q_\alpha} \text{ over } \hat{U}_\alpha \cap \hat{U}_\beta.$$

We can assume  $d^p_\beta$  and  $d^q_\alpha$  are homogeneous of degree  $k$ . Then it's easy to see that  $c^p_{\beta\alpha} = d^p_\beta - d^q_\alpha \frac{\partial w^p_\beta}{\partial w^q_\alpha}$ . So if we define

$$(\tilde{\theta}_k)_{\beta\alpha} = (\theta_k)_{\beta\alpha} - d^p_\beta \frac{\partial}{\partial w^p_\beta} + d^q_\alpha \frac{\partial}{\partial w^q_\alpha}$$

then it's easy to see that  $(\tilde{\theta}_k)_{\beta\alpha} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta, \Theta_{N_S} \otimes \mathcal{I}_S^{k+1})(-k)$  and we have  $\theta_k = \mathfrak{N}'_k(\tilde{\theta}_k)$ .

To show  $\mathfrak{R}_k$  is an isomorphism, we will construct its inverse. Assume  $\mathfrak{h} \in H^1(S, N_S \otimes \mathcal{I}_S^{k+1} / \mathcal{I}_S^{k+2})$ , we can represents it as a cocycle:

$$(\mathfrak{h})_{\beta\alpha} = \sum_{r=1}^m [b^r_{\beta\alpha}]_{k+2} \frac{\partial}{\partial w^r_\beta}.$$

We can assume  $b^r_{\beta\alpha}$  is homogeneous of degree  $k+1$  in  $w'_\beta = \{w^r_\beta\}$ . Then we just define

$$(\mathfrak{h}'_k)_{\beta\alpha} := \mathfrak{R}_k^{-1}((\mathfrak{h})_{\beta\alpha}) = \sum_{r=1}^m b^r_{\beta\alpha} \frac{\partial}{\partial w^r_\beta} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta, \Theta_{N_S} \otimes \mathcal{I}_S^{k+1})(-k).$$

It's easy to see that  $\{(\mathfrak{h}'_k)_{\beta\alpha}\}$  satisfies the cocycle condition and hence represents a cohomology class in  $H^1(N_S, \Theta_{N_S} \otimes \mathcal{I}_S^{k+1})$  of weight  $-k$ .

Using similar homogenization argument, it's also easy to show that  $\mathfrak{J}_k$  is an isomorphism.  $\square$



Our main result in this subsection is the following technical proposition which under appropriate assumption re-interprets the obstructions to splitting and comfortable embeddings via the deformation to normal cone construction:

**Proposition 2.1.** *Assume that  $S$  is  $(k-1)$ -comfortably-embedded submanifold of  $X$  for some  $k \geq 1$  and let  $(\rho_{k-1}, \nu_{k-1})$  be a  $(k-1)$ -comfortable pair. Then there is a reduced Kodaira-Spencer class  $\theta_k \in H^1(N_S, \Theta_{N_S})(-k)$  such that the following relations hold under the exact sequence from Proposition 2.1.*

1.  $\mathfrak{T}_k(\theta_k) = \mathfrak{g}_k^{\rho_{k-1}} \in H^1(S, \Theta_S \otimes \mathcal{I}_S^k / \mathcal{I}_S^{k+1})$  is the obstruction to  $k$ -splitting relative to  $\rho_{k-1}$ . As a consequence, if  $S$  is not  $k$ -splitting relative to  $\rho_{k-1}$ , then  $\theta_k \in H^1(N_S, \Theta_{N_S})(-k)$  is non zero.
2. If  $S$  is  $k$ -splitting relative to  $\rho_{k-1}$ , i.e. we have a  $k$ -th order lifting  $\rho_k$  such that  $\phi_{k,k-1} \circ \rho_k = \rho_{k-1}$ , then  $\theta_k = \mathfrak{N}_k(\mathfrak{h}_k^{\rho_k})$  where  $\mathfrak{h}_k^{\rho_k} \in H^1(S, N_S \otimes \mathcal{I}_S^{k+1} / \mathcal{I}_S^{k+2})$  is the obstruction to  $k$ -comfortably-embedding with respect to  $\rho_k$ .

From now on, we use the same notations such as  $\tilde{\mathcal{X}}$ ,  $V_\alpha$ ,  $\mathcal{U}_\alpha$  and  $\mathcal{U}$  as those in the above section. Then  $\{\mathcal{U}_\alpha \cap \tilde{\mathcal{X}}_0\}$  is an open covering of  $S_0$ . Before we prove Proposition 2.1, we slightly generalize the Kodaira-Spencer class recalled Appendix 5.3. For a differentiable family of (local) complex manifolds  $\tilde{\mathcal{X}}$ , assume that the transition functions are given by

$$z_\alpha^i = f_{\alpha\beta}^i(z_\beta, t), \quad t|_{\mathcal{U}_\alpha} = t|_{\mathcal{U}_\beta} \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta.$$

Assume that the functions  $f_{\alpha\beta}^i(z_\beta, t) - f_{\alpha\beta}^i(z_\beta, 0)$  vanish up to order  $(k-1)$  at  $t=0$ :

$$\left. \frac{\partial^l f_{\alpha\beta}^i(z_\beta, t)}{\partial t^l} \right|_{t=0} = 0, \text{ for } 1 \leq l < k.$$

Using this and the cocycle condition of  $\{f_{\alpha\beta}\}$ , we deduce that:

$$\begin{aligned} f_{\alpha\beta}^i(f_{\beta\gamma}(z_\gamma, t), t) &= f_{\alpha\gamma}^i(z_\gamma, t) \implies \frac{\partial f_{\alpha\beta}^i(z_\beta, t)}{\partial z_\beta^j} \frac{\partial f_{\beta\gamma}^j(z_\gamma, t)}{\partial t} + \frac{\partial f_{\alpha\beta}^i(z_\beta, t)}{\partial t} = \frac{\partial f_{\alpha\gamma}^i(z_\gamma, t)}{\partial t} \\ \implies \frac{\partial f_{\alpha\beta}^i(z_\beta, t)}{\partial z_\beta^j} \frac{\partial f_{\beta\gamma}^j(z_\gamma, t)}{\partial t^k} + O(t) + \frac{\partial^k f_{\alpha\beta}^i(z_\beta, t)}{\partial t^k} &= \frac{\partial^k f_{\alpha\gamma}^i(z_\gamma, t)}{\partial t^k}. \end{aligned}$$

So if we define:

$$(\theta_k)_{\alpha\beta} = \frac{1}{k!} \sum_{i=1}^n \left. \frac{\partial^k f_{\alpha\beta}^i(z_\beta, t)}{\partial t^k} \right|_{t=0} \frac{\partial}{\partial z_\alpha^i} \in H^0(\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \tilde{\mathcal{X}}_0, \Theta_{\mathcal{X}_0}), \quad (15)$$

then  $\theta_k$  satisfies the cocycle condition:

$$(\theta_k)_{\beta\gamma} = (\theta_k)_{\alpha\gamma} - (\theta_k)_{\alpha\beta}. \quad (16)$$

We will call  $\theta_k = \{(\theta_k)_{\alpha\beta}\}$  the reduced Kodaira-Spencer cocycle.

*Proof of Proposition 2.1.* Suppose that the embedding  $S \hookrightarrow X$  is  $(k-1)$ -comfortably embedded. By Theorem 5.5 and the calculations leading to (11), we can choose a  $(k-1)$ -comfortable atlas adapted to  $(\rho_{k-1}, \nu_{k-1})$  such that we have induced atlas on the blow up with transition functions given by:

$$\begin{cases} w_\beta^r &= \sum_{s=1}^m (a_{\beta\alpha})_s^r (w_\alpha'') w_\alpha^s + t^k \tilde{R}_{k+1}^r, & \text{for } r = 1, \dots, m, \\ w_\beta^p &= \phi_{\beta\alpha}^p(w_\alpha'') + t^k \tilde{R}_k^p, & \text{for } p = m+1, \dots, n. \end{cases} \quad (17)$$

We can substitute the transition function in (17) into (15) above to get:

$$(\theta_k)_{\beta\alpha} = \frac{1}{k!} \sum_{i=1}^n \left. \frac{\partial^k f_{\beta\alpha}^i(w_\alpha, t)}{\partial t^k} \right|_{t=0} \frac{\partial}{\partial w_\beta^i} = \sum_{r=1}^m \tilde{R}_{k+1}^r(0; w_\alpha) \frac{\partial}{\partial w_\beta^r} + \sum_{p=m+1}^n \tilde{R}_k^p(0; w_\alpha) \frac{\partial}{\partial w_\beta^p}, \quad (18)$$

where in the last expression,  $w_\alpha$  and  $w_\beta$  are related by the following relation on  $\tilde{\mathcal{X}}_0$  near  $S_0 \cong S$ :

$$\begin{cases} w_\beta^r &= \sum_{s=1}^m (a_{\beta\alpha})_s^r (w_\alpha'') w_\alpha^s, & \text{for } r = 1, \dots, m, \\ w_\beta^p &= \phi_{\beta\alpha}^p(w_\alpha''), & \text{for } p = m+1, \dots, n; \end{cases} \quad (19)$$

which is nothing but the transition function on  $N_S$ . Recall that  $\tilde{R}_{k+1}^r(t; w'_\alpha, w''_\alpha) = t^{-(k+1)} R_{k+1}(tw'_\alpha, w''_\alpha)$  and  $\tilde{R}_k^p(0; w'_\alpha, w''_\alpha) = t^{-k} R_k^p(tw'_\alpha, w''_\alpha)$ . So  $\tilde{R}_{k+1}^r(0; w_\alpha)$  (resp.  $\tilde{R}_k^p(0; w_\alpha)$ ) is nothing but the  $(k+1)$ -th (resp.  $k$ -th) order leading term of  $R_{k+1}^r(w_\alpha)$  (resp.  $R_k^p(w_\alpha)$ ) in its Taylor expansion with respect to  $w'_\alpha$ .

Since  $w'_\alpha$  are global coordinates on the whole  $\hat{U}_\alpha \subset N_S$ , we see that  $(\theta_k)_{\beta\alpha}$  is actually defined over  $\hat{U}_\alpha \cap \hat{U}_\beta \subset N_S$ . So if we denote by  $\pi_S : N_S \rightarrow S$  the natural projection of the normal bundle to its base, and by  $\hat{U}_\alpha = \pi_S^{-1}(\mathcal{U}_\alpha \cap \tilde{\mathcal{X}}_0 \cap S_0)$  the  $\mathbb{C}^*$ -invariant open set on  $N_S$ , then we have:

$$(\theta_k)_{\beta\alpha} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta, \Theta_{N_S} \otimes \mathcal{I}_S^k).$$

So by (16) we get a Čch cohomology class:

$$\theta'_k := \{(\theta_k)_{\beta\alpha}\} \in \check{H}^1(N_S, \Theta_{N_S} \otimes \mathcal{I}_S^k).$$

From (18) and homogeneity of  $\tilde{R}_{k+1}^r, \tilde{R}_k^p$  in  $w'_\alpha$ , we see that  $\theta'_k$  has weight  $(-k)$  under the natural  $\mathbb{C}^*$ -action on  $N_S$ . When we restrict to  $S_0 = S \subset N_S$  and mod-out by  $\mathcal{I}_{S_0}^{k+1}$ , we get:

$$\begin{aligned} (\mathfrak{g}_k)_{\beta\alpha} := (\theta_k)_{\beta\alpha}|_{S_0} &= \sum_{r=1}^m [\tilde{R}_{k+1}^r(0; w'_\alpha, w''_\alpha)]_{(k)} \frac{\partial}{\partial w_\beta^r} + \sum_{p=m+1}^n [\tilde{R}_k^p(0; w'_\alpha, w''_\alpha)]_{(k)} \frac{\partial}{\partial w_\beta^p} \\ &= \sum_{p=m+1}^n [\tilde{R}_k^p(0; w'_\alpha, w''_\alpha)]_{(k)} \frac{\partial}{\partial w_\beta^p}, \end{aligned} \quad (20)$$

which form a cocycle

$$\{(\mathfrak{g}_k)_{\beta\alpha}\} \in \check{H}^1(\{U_\alpha\}, \Theta_{N_S}|_{S_0} \otimes \mathcal{I}_{S_0}^k / \mathcal{I}_{S_0}^{k+1}) = \check{H}^1(\{U_\alpha\}, N_{S_0} \otimes \mathcal{I}_{S_0}^k / \mathcal{I}_{S_0}^{k+1}) \bigoplus \check{H}^1(\{U_\alpha\}, \Theta_{S_0} \otimes \mathcal{I}_{S_0}^k / \mathcal{I}_{S_0}^{k+1}).$$

In the last equality, we used the holomorphic splitting  $\Theta_{N_S}|_{S_0} = \Theta_{S_0} \oplus N_{S_0}$ . Because we assumed that  $S$  is  $(k-1)$ -comfortably-embedded, the component in the first summand is 0 as seen in (20). So using the notation in Lemma 2.1, we can write  $\mathfrak{g}_k = \mathfrak{T}_k(\theta_k)$ . By Theorem 5.1 we see that  $\mathfrak{g}_k = \{(\mathfrak{g}_k)_{\beta\alpha}\}$  is the obstruction to the existence of  $\rho_k$  satisfying  $\phi_{k,k-1} \circ \rho_k = \rho_{k-1}$ . In other words,  $\mathfrak{g}_k^{\rho_{k-1}} := \mathfrak{g}_k$  is the obstruction to  $k$ -splitting relative to  $\rho_{k-1}$ . So we get the first part of Proposition 2.1.

Now if we assume that the obstruction to  $k$ -splitting vanishes, i.e. the above  $\mathfrak{g}_k^{\rho_{k-1}}$  vanishes, then by Theorem 5.6 the transition functions in (17) can be improved to

$$\begin{cases} w_\beta^r &= \sum_{s=1}^m (a_{\beta\alpha})_s^r (w''_\alpha) w_\alpha^s + t^k \tilde{R}_{k+1}^r, & \text{for } r = 1, \dots, m, \\ w_\beta^p &= \phi_{\beta\alpha}^p(w''_\alpha) + t^{k+1} \tilde{R}_{k+1}^p, & \text{for } p = m+1, \dots, n. \end{cases} \quad (21)$$

Substituting this into (15),  $(\theta_k)_{\beta\alpha}$  now becomes:

$$(\theta_k)_{\beta\alpha} = \frac{1}{k!} \sum_{i=1}^n \frac{\partial^k f_{\beta\alpha}^i(w_\alpha, t)}{\partial t^k} \bigg|_{t=0} \frac{\partial}{\partial w_\beta^i} = \sum_{r=1}^m \tilde{R}_{k+1}^r(0; w_\alpha) \frac{\partial}{\partial w_\beta^r}. \quad (22)$$

So we see that in this case  $(\theta_k)_{\beta\alpha} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta, \Theta_{N_S} \otimes \mathcal{I}_S^{k+1})$ . Again by (16), we get a weight  $(-k)$  Čch cohomology class:

$$\theta''_k := \{(\theta_k)_{\beta\alpha}\} \in \check{H}^1(\{\hat{U}_\alpha\}, \Theta_{N_S} \otimes \mathcal{I}_S^{k+1})(-k),$$

which satisfies  $\mathfrak{N}'_k(\theta''_k) = \theta_k$ . When we restrict to  $S_0$  and mod out by  $\mathcal{I}_{S_0}^{k+2}$ , we get:

$$(\mathfrak{h}_k)_{\beta\alpha} := (\theta_k)_{\beta\alpha}|_{S_0} = \sum_{r=1}^m [\tilde{R}_{k+1}^r(0; w'_\alpha, w''_\alpha)]_{(k+1)} \frac{\partial}{\partial w_\beta^r} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta \cap S_0, N_{S_0} \otimes \mathcal{I}_{S_0}^{k+1} / \mathcal{I}_{S_0}^{k+2}). \quad (23)$$

Comparing with (61), we see that  $\mathfrak{h}_k := \{(\mathfrak{h}_k)_{\beta\alpha}\}$  is nothing but the obstruction  $\mathfrak{h}_k^{\rho_k}$  to  $k$ -comfortable embedding with respect to the  $k$ -splitting  $\rho_k$ . By Lemma 2.1, we can write  $\theta''_k = \mathfrak{N}_k^{-1}(\mathfrak{h}_k)$ .  $\square$

### 2.2.2 Case of ample divisors

From now on, we assume  $S = D$  is an ample divisor in  $X$ . Before proving Theorem 1.1, we recall the construction in the introduction and make an important remark. Denote  $\tilde{X} = Bl_{D \times \{0\}}(X \times \mathbb{C})$ . Then  $\tilde{\mathcal{X}}_0 = X \cup E$  where  $E = \mathbb{P}(N_D \oplus \mathbb{C})$ . We assume  $D$  is an ample divisor and denote by  $L := L_D$  the holomorphic line bundle generated by  $D$ , it's easy to see that the line bundle  $\tilde{\mathcal{L}} = \pi^*L - E$  is semi-ample and its linear system  $|m\tilde{\mathcal{L}}|$  for  $m \gg 1$  contracts the component  $X$  of the central fibre  $\tilde{\mathcal{X}}_0$ . In this way we get a degeneration of  $X$  to a singular variety  $\mathcal{X}_0$  which is obtained from  $E$  by contracting the infinity section  $D_\infty$ .

It's well known that the deformation to the normal cone can be realized via graph construction as follows. Let  $s_D$  denote the canonical holomorphic section of  $L = L_D$  with  $D = \{s_D = 0\}$ . We can identify  $X$  with the graph of  $s_D$  as a subvariety of  $Y = \mathbb{P}(L \oplus \mathbb{C})$ :  $\mathcal{X}_1 = \{(p, [s_D(p), 1]); p \in X\}$ . We then use the natural  $\mathbb{C}^*$ -action on  $Y$  to get a family of subvarieties of  $Y$ :  $\mathcal{X}_t = \{(p, [t^{-1}s_D(p), 1]); p \in X\}$ . For  $t \neq 0$ ,  $\mathcal{X}_t \cong X$ . As  $t \rightarrow 0$ ,  $\mathcal{X}_t$  converges to a subscheme  $\tilde{\mathcal{X}}_0$  of  $Y$  which is nothing but the union of  $X$  with  $E$ . See Figure 2 for the illustration. For the contracted deformation to the normal cone, we want to contract  $X$  in the central fibre to get  $\mathcal{X}$  instead of  $\tilde{\mathcal{X}}$ . Then we can use the similar construction, replacing  $Y = \mathbb{P}(L \oplus \mathbb{C})$  by the projective cone  $\overline{C}(X, L)$  which is obtained from  $Y$  by contracting the infinity divisor  $X_\infty$ . Similar discussion appeared in [11, Proof of Proposition 5.1].

$\mathcal{X}_0$  is very close to being the projective cone  $\overline{C}(D, L)$ . One delicate point here is that  $\mathcal{X}_0$  may not be normal.

**Lemma 2.2.** *The central fibre  $\mathcal{X}_0$  coincides with  $\overline{C}(D, L)$  if the restriction map  $\psi_m : H^0(X, mL) \rightarrow H^0(D, mL|_D)$  is surjective for any  $m \geq 0$ .*

*Proof.* Denote by  $H^0(X, L^r)|_D$  the image of the restriction map  $H^0(X, L^r) \rightarrow H^0(D, L^r|_D)$  for any  $r \geq 0$ . Then we can write:

$$\mathcal{X}_0 = \text{Proj} \bigoplus_{m=0}^{\infty} (\oplus_{r=0}^m H^0(X, L^r)|_D \cdot x^{m-r}),$$

where  $x$  has degree 1 in the graded ring. On the other hand, we have:

$$\overline{C}(D, L) = \text{Proj} \bigoplus_{m=0}^{\infty} (\oplus_{r=0}^m H^0(D, L^r|_D) \cdot x^{m-r}).$$

So we see that  $\mathcal{X}_0 \cong \overline{C}(D, L)$  if  $H^0(X, L^r)|_D = H^0(D, L^r|_D)$ . □

For example, let  $X$  be any Riemann surface of genus  $\geq 1$ .  $D = \{p\}$  is any point. Then  $D$  is ample. In this special case, the central fibre  $\mathcal{X}_0$  is a singular curve whose normalization is  $\mathbb{P}^1$ . Here the map  $\psi_0 = id : H^0(X, \mathcal{O}_X) \rightarrow H^0(p, \mathcal{O}_p)$ . But  $\psi_1 = 0 : H^0(X, L_p) = \mathbb{C} \rightarrow H^0(\{p\}, L_p|_{\{p\}}) = \mathbb{C}$  because  $\psi_1$  factors through the inverse of isomorphism  $H^0(X, \mathcal{O}_X) = \mathbb{C} \xrightarrow{\cdot s_{\{p\}}} H^0(X, L_p)$  by the assumption that  $g(X) \geq 1$ . In particular,  $\psi_1$  is not surjective.

On the other hand, from the exact sequence:

$$0 \rightarrow H^0(X, (m-1)L_D) \rightarrow H^0(X, mL_D) \rightarrow H^0(D, mL|_D) \rightarrow H^1(X, (m-1)L) \rightarrow \cdots,$$

we see that  $\psi_m$  is surjective if  $H^1(X, (m-1)L) = 0$  for all  $m \geq 1$ . In particular, this is satisfied in the Tian-Yau setting. Indeed, if  $X$  is Fano and  $m \geq 1$  then  $H^1(X, (m-1)L) = H^1(X, \Omega_X^n \otimes \mathcal{O}_X(-K_X + (m-1)L)) = 0$  by the Nakano-Kodaira vanishing theorem.

From now on, we assume that we are in the situation that central fiber  $\mathcal{X}_0$ , coming from the contracted deformation to the normal cone, is normal and hence coincides with  $\overline{C}(D, L)$ . Combining the calculations from the previous subsection and the above discussion, we can derive the following

**Proposition 2.2.** *Let  $D \hookrightarrow X$  be a  $(k-1)$ -comfortably embedded and  $(\rho_{k-1}, \nu_{k-1})$  be a  $(k-1)$ -comfortable pair. If  $D$  is not  $k$ -splitting relative to  $\rho_{k-1}$ , then  $w(X, D) = k$ . In particular, if  $D$  is  $(k-1)$ -comfortably embedded and not  $k$ -splitting, then  $w(X, D) = k$ .*

*Proof.* Let  $(\rho_{k-1}, \nu_{k-1})$  be a  $(k-1)$ -comfortable pair. Then from Proposition 2.1, we get a cohomology class  $\theta_k \in H^1(L, \Theta_L)$  with weight  $-k$ , which is represented by a cocycle  $\{(\theta_k)_{\beta\alpha}\}$ . Recall that we defined the reduced Kodaira-Spencer in the introduction:

$$\mathbf{KS}_{\mathcal{X}^\circ}^{(j)} = \frac{1}{j!} \frac{d^j}{dt^j} \mathbf{I}_{\mathcal{X}^\circ} \Big|_{t=0} \in \mathbf{T}_C^1.$$

From the construction and expression of  $\theta_k$  (see (17)-(18)) in the proof of Proposition 2.1, we know that  $\mathbf{I}_{\mathcal{X}^\circ}$  vanishes up to order  $k-1$  and we have:

$$\mathbf{KS}_{\mathcal{X}^\circ}^{(k)} = [\theta^{(k)}] \Big|_U =: \vartheta^{(k)} \in \mathbf{T}_C^1(-k) \subset H^1(U, \Theta_U)(-k).$$

The last inclusion is by Schlessinger's result in [27] (see Appendix 5.4). To finish the proof, we just need to show that  $\vartheta^{(k)} \neq 0 \in H^1(U, \Theta_U)(-k)$ .

By Proposition 2.1, we know that  $\mathfrak{T}_k(\theta_k) = \mathfrak{g}_k$  is the obstruction to  $k$ -splitting relative to  $\rho_{k-1}$ . So if the embedding is not  $k$ -splitting with respect to  $\rho_{k-1}$ , then  $\theta_k$  is non zero. Now the claim follows from the Lemma 2.3.  $\square$

**Corollary 2.1.** *Assume  $\dim D = n-1 \geq 2$ . If  $D$  is  $(k-1)$ -comfortably embedded and not  $k$ -comfortably embedded, then the following holds:*

1.  $w(X, D) = -k$ , i.e. Theorem 1.1 is true;
2. For any  $l < k$  and any  $(l-1)$ -th order lifting  $\rho_{l-1} : \mathcal{O}_D \rightarrow \mathcal{O}_X/\mathcal{I}_D^l$ , there exists a  $(k-1)$ -th order lifting  $\rho_{k-1} : \mathcal{O}_D \rightarrow \mathcal{O}_X/\mathcal{I}_D^k$  such that  $\phi_{k-1, l-1}(\rho_{k-1}) = \rho_{l-1}$ , where  $\phi_{k-1, l-1} : \mathcal{O}_X/\mathcal{I}_D^k \rightarrow \mathcal{O}_X/\mathcal{I}_D^l$  is the natural map.

*Proof.* We first recall Remark 5.4. If  $\dim D \geq 2$  and  $D$  is ample,  $H^1(D, N_D \otimes \mathcal{I}_D^{k+1}/\mathcal{I}_D^{k+2}) = H^1(D, L_D^{-k}) = 0$  for any  $k \geq 1$  by Kodaira-Nakano vanishing. So there is no obstruction to  $k$ -comfortably embedded relative to any  $k$ -th order lifting, and so  $k$ -comfortable is equivalent to  $k$ -splitting for any  $k \geq 0$ .

Hence by the assumption, we know that  $(X, D)$  is  $(k-1)$ -splitting but not  $k$ -splitting. So there exists a comfortable pair  $(\rho_{k-1}, \nu_{k-1})$  such that there is no  $k$ -th order lifting relative to  $\rho_{k-1}$ . By Proposition 2.2, we know that  $w(X, D) = -k$ .

Suppose that for some  $l < k$ , there exists an  $(l-1)$ -th order lifting  $\rho_{l-1}$  that can not be lifted to a  $(k-1)$ -order lifting. By choosing the maximal  $l$  and using Remark 5.4, we can assume there is a comfortable pair  $(\rho_{l-1}, \nu_{l-1})$  such that  $\rho_{l-1}$  can not be lifted to an  $l$ -th order lifting. By Proposition 2.2, we get  $w(X, D) = -l > -k$  which contradicts part 1.  $\square$

**Remark 2.1.** *We will see in Proposition 2.3 that part 2 of the Corollary 2.1 is not necessarily true if  $n = 2$ .*

**Lemma 2.3.** *For  $k \geq 1$ , the natural restriction map  $H^1(L, \Theta_L)(-k) \rightarrow H^1(U, \Theta_U)(-k)$  is an isomorphism.*

*Proof.* This is already clear by the homogenization argument as in the proof of Corollary 2.1. For comparison, we also give a slightly conceptual proof. On the total space  $L$ , we have the exact sequence:

$$0 \rightarrow \pi_L^* L \rightarrow \Theta_L \rightarrow \pi_L^* \Theta_D \rightarrow 0. \quad (24)$$

By restricting this exact sequence to  $U = L \setminus D$ , we have a similar exact sequence on  $U$ . So we get commutative diagram of long exact sequences:

$$\begin{array}{ccccccccc} H^0(L, \pi_L^* \Theta_D) & \rightarrow & H^1(L, \pi_L^* L) & \rightarrow & H^1(L, \Theta_L) & \rightarrow & H^1(L, \pi_L^* \Theta_D) & \rightarrow & H^2(L, \pi_L^* L) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(U, \pi_U^* \Theta_D) & \rightarrow & H^1(U, \pi_U^* L) & \rightarrow & H^1(U, \Theta_U) & \rightarrow & H^1(U, \pi_U^* \Theta_D) & \rightarrow & H^2(U, \pi_U^* L) \end{array} \quad (25)$$

By projection formula we have (note the range of indices)

$$\begin{aligned} H^p(L, \pi_L^* L) &= \bigoplus_{j=-1}^{+\infty} H^p(D, L^{-j}), & H^p(L, \pi_L^* \Theta_D) &= \bigoplus_{j=0}^{+\infty} H^p(D, \Theta_D \otimes L^{-j}). \\ H^p(U, \pi_U^* L) &= \bigoplus_{j=-\infty}^{+\infty} H^p(D, L^{-j}), & H^p(U, \pi_U^* \Theta_D) &= \bigoplus_{j=-\infty}^{+\infty} H^p(D, \Theta_D \otimes L^{-j}). \end{aligned}$$

For  $k \geq 1$ , we can extract the weight  $(-k)$ -part to get the exact sequences:

$$\begin{array}{ccccccc} H^0(D, \Theta_D \otimes L^{-k}) & \rightarrow & H^1(D, L^{-k}) & \xrightarrow{\mathfrak{N}_k} & H^1(L, \Theta_L)(-k) & \xrightarrow{\mathfrak{T}_k} & H^1(D, \Theta_D \otimes L^{-k}) \rightarrow H^2(D, L^{-k}) \\ \parallel & & \parallel & & \downarrow & & \parallel \\ H^0(D, \Theta_D \otimes L^{-k}) & \rightarrow & H^1(D, L^{-k}) & \xrightarrow{\mathfrak{N}_k^\circ} & H^1(U, \Theta_U)(-k) & \xrightarrow{\mathfrak{T}_k^\circ} & H^1(D, \Theta_D \otimes L^{-k}) \rightarrow H^2(D, L^{-k}) \end{array} \quad (26)$$

Now the claim follows from the 5-lemma.  $\square$

**Remark 2.2.** The above proof fits Proposition 2.1 in the following sense. Under the short exact sequence:

$$H^1(D, L^{-k}) \xrightarrow{\mathfrak{N}_k^\circ} H^1(U, \Theta_U)(-k) \xrightarrow{\mathfrak{T}_k^\circ} H^1(D, \Theta_D \otimes L^{-k}) \quad (27)$$

we have: 1.  $\mathfrak{T}_k^\circ(\vartheta^{(k)}) = \mathfrak{g}_k$  is the obstruction to  $k$ -splitting; 2. If  $\mathfrak{T}_k^\circ(\vartheta^{(k)}) = 0$ , then there is a  $k$ -th order lifting  $\rho_k$  and  $\vartheta^{(k)} = \mathfrak{N}_k^\circ(\mathfrak{h}^{(k)})$  where  $\mathfrak{h}^{(k)}$  is the obstruction to  $k$ -comfortably-embedding relative to  $\rho_k$ .

## 2.3 2-dimensional examples and a remark on comfortable embedding

As mentioned in the introduction and recalled in Appendix 5.2, in [1], the authors gave a detailed study of various conditions of embedding:  $k$ -linearizable,  $k$ -splitting and  $k$ -comfortable embedding. In order to talk about  $k$ -comfortable embedding, one needs to assume  $k$ -splitting (see Definition 5.3). Under this assumption, we can study whether the embedding is comfortable with respect to any  $k$ -th order lifting. In [1, Remark 3.4], the authors asked whether  $k$ -comfortable embedding with respect to one  $k$ -th order lifting implies  $k$ -comfortable embedding with respect to any other  $k$ -th order lifting. Here we give a simple example showing that the answer to this question is in general negative.

**Proposition 2.3.** The following is true for the diagonal embedding  $D = \Delta(\mathbb{P}^1) \hookrightarrow X = \mathbb{P}^1 \times \mathbb{P}^1$ :

- (i) It is  $k$ -splitting for any  $k \geq 1$ .
- (ii) The set of all 1st order liftings is parametrized by  $\mathbb{C}$ . So we can denote by  $\rho_1^a$  the 1st order lifting corresponding to any  $a \in \mathbb{C}$ .
- (iii) There exists a 2nd order lifting  $\rho_2$  satisfying  $\phi_{2,1} \circ \rho_2 = \rho_1^a$  if and only if  $a = 0$ .
- (iv) The embedding is 1-comfortable with respect to  $\rho_1^a$  if and only if  $a = -1/2$ .
- (v) The embedding is 1-linearizable but not 2-linearizable.

**Remark 2.3.** This diagonal embedding is 2-splitting and 1-comfortable, but the embedding is only 1-linearizable. This does not contradict Theorem 5.6, since the 1-comfortable embedding is with respect to  $\rho_1^{-1/2}$  which can not be lifted to a 2nd order lifting.

*Proof.* Because there is a surjective morphism  $p : X = D \times D \rightarrow D$ , we see that there is a natural  $k$ -th order lifting  $\rho_k : \mathcal{O}_D \rightarrow \mathcal{O}_X/\mathcal{I}_D^{k+1}$  given by  $\phi_{\infty,k} \circ p^*$ , where  $p^* : \mathcal{O}_D \rightarrow \mathcal{O}_X$  is the pull-back and  $\phi_{\infty,k} : \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{I}_D^{k+1}$  is the natural quotient map. So the embedding is  $k$ -splitting for any  $k \geq 1$ . Since any embedding is 0-comfortable, we know that the embedding is 1-linearizable by Theorem 5.6. So we get (i) and first half of (v).

We will quickly show that the the embedding is not comfortable with respect to the natural 1st order lifting  $\rho_1$ . We first construct an atlas near  $D$ . Choose the open covering of  $\mathbb{P}^1 \times \mathbb{P}^1$ :

$$\mathfrak{V} = \{U_i \times U_j; 1 \leq i, j \leq 2\}.$$

with (we denote  $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$  with  $|\infty| = +\infty$ )

$$U_1 = \{z \in \mathbb{P}^1; |z| < 2\}, U_2 = \{z \in \mathbb{P}^1; |z| > 1/2\}.$$

Then  $S = \Delta(\mathbb{P}^1)$  is covered by two open sets  $\{V_i := U_i \times U_i; i = 1, 2\}$ , we define new coordinate functions by:

$$\begin{aligned} V_1 &= \{(z, z') \in \mathbb{P}^1 \times \mathbb{P}^1; |z| < 2, |z'| < 2\} \rightarrow \mathbb{C}^2 \\ &\quad (z, z') \mapsto (y_1 = z - z', z_1 = z) \\ V_2 &= \{(z, z') \in \mathbb{P}^1 \times \mathbb{P}^1; |z| > 1/2, |z'| > 1/2\} \rightarrow \mathbb{C}^2 \\ &\quad (z, z') \mapsto (y_2 = z^{-1} - z'^{-1}, z_2 = z^{-1}). \end{aligned}$$

So we have  $D \cap V_i = \{y_i = 0\}$ . If  $V'$  is a small neighborhood of  $S = \Delta(\mathbb{P}^1)$  Then on the intersection  $V_1 \cap V_2 \cap V'$ , the transition functions are given by:

$$y_2 = -\frac{y_1}{z_1(z_1 - y_1)} = -\frac{y_1}{z_1^2} - \frac{y_1^2}{z_1^3} + R_3, \quad z_2 = z_1^{-1}. \quad (28)$$

In the above expansion, we assume that  $y_1$  is sufficiently small, and denote by  $R_3$  a term  $\in \mathcal{I}_D^3$ . It's immediate to see that this atlas is adapted to the natural 1st order lifting  $\rho_1$  where we have:

$$\rho_1(z_1) = [z_1]_2 \text{ on } V_1 \cap V', \quad \rho_1(z_2) = [z_2]_2 \text{ on } V_2 \cap V'.$$

The obstruction to 1-comfortable embedding is given by

$$(\mathfrak{h}_1^{\rho_1})_{21} = -\frac{[y_1^2]_3}{z_1^3} \frac{\partial}{\partial y_2} \in H^0(U_1 \cap U_2, N_D \otimes \mathcal{I}_D^2/\mathcal{I}_D^3). \quad (29)$$

Here we consider  $\frac{\partial}{\partial y_2}$  and  $\frac{\partial}{\partial y_1}$  as local generators of  $N_D$ , so that we have  $\frac{\partial}{\partial y_2} = -z_1^2 \frac{\partial}{\partial y_1}$  on  $U_1 \cap U_2$ . We claim that  $\mathfrak{h}_1^{\rho_1}$  represents a nonzero cohomology class in  $H^1(D, N_D \otimes \mathcal{I}_D^2/\mathcal{I}_D^3) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = \mathbb{C}$ . Otherwise, we can write:

$$-\frac{[y_1^2]_3}{z_1^3} \frac{\partial}{\partial y_2} = a[y_1^2]_3 \frac{\partial}{\partial y_1} - b[y_2^2]_3 \frac{\partial}{\partial y_2} \text{ on } U_1 \cap U_2$$

where  $a = a(z_1)$  is analytic in  $z_1$  and  $b = b(z_1^{-1})$  is analytic in  $z_2 = z_1^{-1}$ . Using the change of coordinates, we arrive at an equation:

$$-\frac{1}{z_1} = a(z_1) - \frac{b(z_1^{-1})}{z_1^2},$$

which obviously has no solutions by looking at the Laurent expansion. So we get that  $D \hookrightarrow X$  is not 1-comfortably embedded with respect to  $\rho_1$ .

Let's find all possible 1st order liftings, i.e. homomorphisms of sheaves of rings  $\rho : \mathcal{O}_D = \mathcal{O}_X/\mathcal{I}_D \rightarrow \mathcal{O}_X/\mathcal{I}_D^2$  with  $\phi_{1,0} \circ \rho = id$ . On  $U_1$ , we can write  $\rho(z_1) = [z_1 + a(z_1)y_1]_2$  with  $a(z_1)$  analytic in  $z_1$  and  $\rho(z_2) = [z_2 + b(z_2)y_2]_2$  with  $b(z_2)$  analytic in  $z_2 = z_1^{-1}$ . Since  $\rho$  is a homomorphism of sheaves of rings, we must have

$$1 = \rho(z_1 z_2) = [z_1 z_2 + a(z_1)z_1 y_1 + b(z_2)z_2 y_2]_2 = 1 + a(z_1)z_1[y_1]_2 + b(z_2)z_2[y_2]_2 \text{ over } U_1 \cap U_2.$$

Since we have  $[y_2]_2 = -[y_1]_2 z_1^{-2}$  by (28), we get  $(a(z_1) - b(z_2))z_1[y_1]_2 = 0$ . So we must have that  $a(z_1) = b(z_2) = a = \text{constant}$ . Thus we get (ii). We will denote the corresponding 1st order lifting by  $\rho^a$ .

Now for any fixed 1st order lifting  $\rho^a$ , it's easy to find an atlas adapted to it. We simply need to make a coordinate change:

$$\hat{z}_1 = z_1 + ay_1, \hat{y}_1 = y_1 \text{ on } V_1; \quad \hat{z}_2 = z_2 + ay_2, \hat{y}_2 = y_2 \text{ on } V_2.$$

We can calculate the new transition function:

$$\hat{y}_2 = -\frac{\hat{y}_1}{\hat{z}_1^2} - (2a+1)\frac{\hat{y}_1^2}{\hat{z}_1^3} + R_3, \quad \hat{z}_2 = \hat{z}_1^{-1} - \frac{a^2 \hat{y}_1^2}{\hat{z}_1^3} + R_3,$$

where  $R_3$  denotes terms in  $\mathcal{I}_D^3$ . So we see that the obstruction to 1-comfortable embedding with respect to  $\rho_1^a$  is equal to  $(2a+1)\mathfrak{h}_1^{\rho_1^a}$  (see (29)). From above we have seen that  $H^1(D, N_D \otimes \mathcal{I}_D^2/\mathcal{I}_D^3) \cong \mathbb{C}$  is generated by  $\mathfrak{h}_1^{\rho_1^a}$ . So the embedding is comfortable with respect to  $\rho_1^a$  if and only if  $a = -1/2$ . So we get (iii).

Furthermore, we can calculate the obstruction to existence of 2nd order lifting  $\rho_2^a$  such that  $\phi_{2,1} \circ \rho_2^a = \rho_1^a$ :

$$\left(\mathfrak{g}_2^{\rho_1^a}\right)_{21} = -a^2 \frac{[y_1^2]_3}{z_1^3} \frac{\partial}{\partial z_2} \in H^0(U_1 \cap U_2, \Theta_D \otimes \mathcal{I}_D^2/\mathcal{I}_D^3).$$

By similar reasoning as before, we can see that  $H^1(D, \Theta_D \otimes \mathcal{I}_D^2/\mathcal{I}_D^3) = H^1(\mathbb{P}^1, \Theta_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(-4)) \cong \mathbb{C}$  is generated by the cohomology  $\mathfrak{g}_2^{\rho_1^a}$  if and only if  $a \neq 0$ . So we get (iv).

If the embedding is 2-linearizable, then it is 2-splitting and 1-comfortably with respect to the induced 1-splitting (see Theorem 5.6). But from (ii)-(iv), we see that no such kind of 1-splitting exists. So we get second half of (v).  $\square$

**Remark 2.4.** By Theorem 5.2, 1-comfortable embedding is equivalent to the splitting of the exact sequence:

$$0 \rightarrow \mathcal{I}_D^2/\mathcal{I}_D^3 \rightarrow \mathcal{I}_D/\mathcal{I}_D^3 \rightarrow \mathcal{I}_D/\mathcal{I}_D^2 \rightarrow 0. \quad (30)$$

This is a priori sequence of sheaves of  $\mathcal{O}_X/\mathcal{I}_D^2$ -modules.  $\mathcal{I}_D^2/\mathcal{I}_D^3$  and  $\mathcal{I}_D/\mathcal{I}_D^2$  are natural  $\mathcal{O}_D$ -modules.  $\mathcal{I}_D/\mathcal{I}_D^3$  becomes a  $\mathcal{O}_D$ -module depending on the 1st order lifting (ring homomorphism)  $\rho_1^a : \mathcal{O}_D \rightarrow \mathcal{O}_X/\mathcal{I}_D^2$ .

(iv) in Proposition 2.3 is equivalent to saying that (30) splits as an exact sequence of  $\mathcal{O}_D$ -modules thus obtained if and only if  $a = -1/2$ . This can also be verified directly using the expression:  $\rho_1^a(z_1) = [z_1 + ay_1]_2$  on  $U_1$  and  $\rho_1^a(z_2) = [z_2 + ay_2]_2$  on  $U_2$ .

**Remark 2.5.** If we denote by  $w_i$  the fiber variables of  $N_D$  satisfying  $w_2 = -z_1^{-2}w_1$ , then using the notation in Lemma 2.1, we have:  $\theta_1 = \mathfrak{Y}_1(\mathfrak{h}_1^{\rho_1}) = 0$  and  $\mathfrak{T}_2(\theta_2^a) = \mathfrak{g}_2^{\rho_1^a}$ , where

$$(\theta_1)_{21} = -\frac{w_1^2}{z_1^3} \frac{\partial}{\partial w_2} = \frac{1}{2}w_2 \frac{\partial}{\partial z_2} - \frac{1}{2}w_1 \frac{\partial}{\partial z_1} \in H^0(\hat{U}_1 \cap \hat{U}_2, \Theta_{N_D})(-1),$$

and

$$(\theta_2^a)_{21} = -(2a+1) \frac{w_1^2}{z_1^3} \frac{\partial}{\partial w_2} - a^2 \frac{w_1^2}{z_1^3} \frac{\partial}{\partial z_2} \in H^0(\hat{U}_1 \cap \hat{U}_2, \Theta_{N_D})(-2).$$

Notice that the central fiber of  $\mathcal{X}$  from the contracted deformation to the normal cone is  $\overline{C}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \cong \mathbb{P}(1, 1, 2)$ . So by Proposition 2.2, we get the following corollary (See Example 3.1 and Remark 3.2).

**Corollary 2.2.** *The contracted deformation to the normal cone associated with  $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta(\mathbb{P}^1))$  degenerates  $\mathbb{P}^1 \times \mathbb{P}^1$  to  $\mathbb{P}(1, 1, 2)$ . The weight of this deformation is  $-2$ .*

Similarly we can deal with the case  $D_2 = \{Z_0^2 + Z_1^2 + Z_2^2 = 0\} \hookrightarrow X_2 = \mathbb{P}^2$ . For this, we notice that there is a 2-fold branched covering:

$$\begin{aligned} p_2 : \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^2 \\ ([X_0, X_1], [Y_0, Y_1]) &\mapsto [X_0 Y_0 + X_1 Y_1, \sqrt{-1}(X_0 Y_0 - X_1 Y_1), \sqrt{-1}(X_0 Y_1 + X_1 Y_0)]. \end{aligned}$$

The branch locus is exactly  $\Delta(\mathbb{P}^1)$  with  $p_2(\Delta(\mathbb{P}^1)) = D_2$ . Using this covering structure, it's easy to obtain two open sets  $\{V_1, V_2\}$  covering  $D_2$ .

$$\begin{aligned} V_1 = (U_1 \times U_1)/\mathbb{Z}^2 &\rightarrow \mathbb{C}^2 \\ (z, z') &\mapsto (y_1 = \frac{1}{4}(z - z')^2, z_1 = \frac{1}{2}(z + z')) \\ V_2 = (U_2 \times U_2)/\mathbb{Z}^2 &\rightarrow \mathbb{C}^2 \\ (z, z') &\mapsto (y_2 = \frac{1}{4}(z^{-1} - z'^{-1})^2, z_2 = \frac{1}{2}(z^{-1} + z'^{-1})) \end{aligned}$$

The transition function over  $V_1 \cap V_2$  is given by:

$$y_2 = \frac{y_1}{(z_1^2 - y_1)^2} = \frac{y_1}{z_1^4} + \frac{2y_1^2}{z_1^6} + R_3, \quad z_2 = \frac{z_1}{z_1^2 - y_1} = \frac{1}{z_1} + \frac{y_1}{z_1^3} + R_2.$$

So this atlas is a 0-comfortable one. The associated  $\theta_1 \in H^1(D_2, N_{D_2})(-1)$  is represented by

$$(\theta_1)_{21} = \frac{2w_1^2}{z_1^6} \frac{\partial}{\partial w_2} + \frac{w_1}{z_1^3} \frac{\partial}{\partial z_2} \in H^0(\hat{U}_1 \cap \hat{U}_2, \Theta_{N_{D_2}})$$

where  $w_i$  are fiber variables of  $N_{D_2} \cong \mathcal{O}_{\mathbb{P}^1}(4)$  satisfying  $w_2 = z_1^{-4}w_1$ . So we have

$$(\mathfrak{g}_1)_{21} = (\mathfrak{T}_1(\theta_1))_{21} = \frac{[w_1]_2}{z_1^3} \frac{\partial}{\partial z_2} \in H^0(U_1 \cap U_2, \Theta_{D_2} \otimes \mathcal{I}_{D_2}/\mathcal{I}_{D_2}^2).$$

In the Čech cohomology  $\check{H}^1(\{U_1, U_2\}, \Theta_{D_2} \otimes \mathcal{I}_{D_2}/\mathcal{I}_{D_2}^2)$ , any coboundary can be represented by

$$a(z_1)[w_1]_2 \frac{\partial}{\partial z_1} - b(z_2)[w_2]_2 \frac{\partial}{\partial z_2} = \left( \frac{-a(z_1)}{z_1^2} - \frac{b(z_1^{-1})}{z_1^4} \right) [w_1]_2 \frac{\partial}{\partial z_2}.$$

Since  $a(z_1)$  (resp.  $b(z_1^{-1})$ ) is analytic in  $z_1$  (resp.  $z_1^{-1}$ ), the term in the bracket of the right hand side can not contain any  $z_1^{-3}$ -term. So we see that  $H^1(D_2, \Theta_{D_2} \otimes \mathcal{I}_{D_2}/\mathcal{I}_{D_2}^2) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong \mathbb{C}$  is generated by  $\mathfrak{g}_1 \neq 0$ . Because  $\mathfrak{g}_1$  is the obstruction to 1-splitting (Theorem 5.1), we obtain that the embedding is not even 1-splitting and hence not 1-linearizable. In this case,  $\mathcal{X}_0 = \overline{C}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)) \cong \mathbb{P}(1, 1, 4)$ . So by Proposition 2.2, we obtain the following result now.

**Proposition 2.4.**  $D_2 = \{Z_0^2 + Z_1^2 + Z_2^2 = 0\} \hookrightarrow \mathbb{P}^2$  is 0-linearizable. The contracted deformation to normal cone associated to  $(\mathbb{P}^2, D_2)$  degenerates  $\mathbb{P}^2$  to  $\mathbb{P}(1, 1, 4)$ . The deformation weight  $w(X, D)$  is equal to  $-1$ .



### 3 Applications to AC Kähler metrics

#### 3.1 Rotationally symmetric Kähler metric on the cone

We consider the Kähler metric on  $C(D, L)$  given by the special Calabi ansatz  $\omega_0 = \sqrt{-1}\partial\bar{\partial}h^\delta$ . Then  $\omega_0$  is a Riemannian cone metric on  $C(D, L)$ :

$$g = dr^2 + r^2 g_Y,$$

where  $Y$  is the associated circle bundle over  $D$ . To see this, we consider the coordinate chart on  $\mathbb{P}(L^{-1} \oplus \mathbb{C})$ . Away from the infinity section  $D_\infty$ , we have coordinate chart given by  $(z, [\zeta_\alpha e_\alpha, 1]) = (z, [e_\alpha, \zeta_\alpha^{-1}]) = (z, [e_\alpha, \xi_\alpha])$ . Let  $h = |e_\alpha|^2 |\zeta_\alpha|^2 = a_{\alpha-}(z) |\zeta_\alpha|^2 = (a_{\alpha+}(z) |\xi_\alpha|^2)^{-1}$ . For simplicity, we will denote  $\zeta = \zeta_\alpha$ ,  $\xi = \xi_\alpha$ ,  $a = a_{\alpha-} = a_{\alpha+}^{-1}$ . Then we can calculate:

$$\omega_0 = \sqrt{-1}\partial\bar{\partial}h^\delta = \delta h^\delta \omega_D + \delta^2 h^\delta \frac{\nabla \zeta \wedge \bar{\nabla} \zeta}{|\zeta|^2} = \delta h^\delta \omega_D + \delta^2 h^\delta \frac{\nabla \xi \wedge \bar{\nabla} \xi}{|\xi|^2}. \quad (31)$$

where  $\omega_D = \sqrt{-1}\partial\bar{\partial} \log h$  is a smooth Kähler metric on  $D$ , and we have used vertical and horizontal frames:

$$dz^i, \nabla \zeta = d\zeta + \zeta a^{-1} \partial a \quad \xleftrightarrow{\text{dual}} \quad \nabla_{z^i} = \frac{\partial}{\partial z^i} - a^{-1} \frac{\partial a}{\partial z^i} \zeta \frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \bar{\zeta}}.$$

Under the  $\{z, \xi\}$  coordinate, we have similarly:

$$dz^i, \nabla \xi = d\xi - \xi a^{-1} \partial a = -\zeta^{-2} \nabla \zeta \quad \xleftrightarrow{\text{dual}} \quad \nabla_{z^i} = \frac{\partial}{\partial z^i} + a^{-1} \frac{\partial a}{\partial z^i} \xi \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \bar{\xi}} = -\zeta^2 \frac{\partial}{\partial \bar{\zeta}}.$$

To write the metric into a metric cone, we write  $\zeta = \tilde{\rho} e^{i\theta}$ . Then

$$\nabla \zeta = d\zeta + \zeta a^{-1} \partial a = e^{i\theta} (d\tilde{\rho} + i\tilde{\rho} d\theta + \tilde{\rho} a^{-1} \partial a) = e^{i\theta} (d\tilde{\rho} + i\tilde{\rho} (d\theta - ia^{-1} \partial a)).$$

So if we let  $r = h^{\delta/2} = (a(z) |\zeta|^2)^{\delta/2}$  and  $\nabla \theta = d\theta - ia^{-1} \partial a$ , then it's easy to verify that the corresponding metric tensor is:

$$g_{\omega_0} = dr^2 + r^2 (\delta g_{\omega_D} + \delta^2 \nabla \theta \otimes \bar{\nabla} \theta).$$

Note that  $\nabla \theta$  is nothing but the connection form on the unit  $S^1$ -bundle in  $L^{-1}$ . Now we compare the norm of tensors on  $U = L \setminus D$  with respect to two metrics  $\omega_0$  and  $\tilde{\omega}_0$ , where  $\tilde{\omega}_0$  is any smooth Kähler metric on a neighborhood of  $D$  in  $L$ . For example, we can take

$$\tilde{\omega}_0 = \pi_L^* \omega_D + \epsilon \sqrt{-1} \partial \bar{\partial} (a_+(z) |\xi|^2)$$

for small  $\epsilon > 0$ . Suppose  $\Phi$  is a tensor of type  $(p = p_h + p_v, q = q_h + q_v)$ , i.e.

$$\Phi \in (T_h^* X)^{\otimes p_h} \otimes (T_v^* X)^{\otimes p_v} \otimes (T_h X)^{\otimes q_h} \otimes (T_v X)^{\otimes q_v}.$$

Then, by noticing  $h^{\delta/2} \sim |\xi|^{-\delta}$ , we have

$$\frac{|\Phi|_{\omega_0}}{|\Phi|_{\tilde{\omega}_0}} \sim |\xi|^{\delta p_h + (\delta+1)p_v - \delta q_h - (\delta+1)q_v}. \quad (32)$$

In particular, we get :

**Lemma 3.1.** *If  $\Phi$  is tensor of type  $(1, 1)$ , then*

$$|\Phi_v^h|_{\omega_0} \sim |\Phi_v^h|_{\tilde{\omega}_0} |\xi|, \quad |\Phi_h^v|_{\omega_0} \sim |\Phi_h^v|_{\tilde{\omega}_0} |\xi|^{-1}, \quad |\Phi_v^v|_{\omega_0} \sim |\Phi_v^v|_{\tilde{\omega}_0}, \quad |\Phi_h^h|_{\omega_0} = |\Phi_h^h|_{\tilde{\omega}_0}.$$

So if  $|\Phi|_{\omega_0} \sim |\xi|^\eta$ , then we have

$$|\Phi_v^h|_{\tilde{\omega}_0} \sim |\xi|^{\eta-1}, \quad |\Phi_h^v|_{\tilde{\omega}_0} \sim |\xi|^{\eta+1}, \quad |\Phi_v^v|_{\tilde{\omega}_0} \sim |\xi|^\eta, \quad |\Phi_h^h|_{\tilde{\omega}_0} \sim |\xi|^\eta. \quad (33)$$

Next we compare the Christoffel symbols of the two metrics, which will be useful for converting the estimate with respect to  $\omega_0$  to that with respect to  $\tilde{\omega}_0$ . See (37)-(38). To simplify the calculation, we can choose the coordinate  $\{z_\alpha^i\}$  on  $D$  and holomorphic frame such that

$$g_{i\bar{j}}^D(0) = \omega_D(\partial_{z_\alpha^i}, \partial_{z_\alpha^j})(0) = \delta_{ij}, (\partial_{z_\alpha^k} g_{i\bar{j}}^D)(0) = 0; \quad (\partial_{z_\alpha^i} a)(0) = 0, (\partial_{z_\alpha^i} \partial_{z_\alpha^j} a)(0) = 0.$$

Denote by the index 0 the coordinate corresponding to  $\xi = \xi_\alpha$ , we then have the components of the metric tensor associated with  $\omega_0$ :

$$g_{i\bar{j}} = \delta a^\delta |\xi|^{-2\delta} \delta_{ij}, g_{0\bar{0}} = \delta^2 a^\delta |\xi|^{-2(\delta+1)}, g_{0\bar{j}} = g_{j\bar{0}} = 0.$$

So it's easy to calculate that:

$$|dz_\alpha^i|_{\omega_0} = \delta^{-1/2} a^{-\delta/2} |\xi|^\delta \sim \frac{1}{|\xi|^{-\delta}}, \quad |d\xi|_{\omega_0} = \delta^{-1} a^{-\delta/2} |\xi|^{(\delta+1)} \sim \frac{|\xi|}{|\xi|^{-\delta}}.$$

$$\Gamma_{ij}^k = \Gamma_{ij}^0 = \Gamma_{i0}^0 = \Gamma_{00}^i = 0, \quad \Gamma_{i0}^j = -\frac{\delta}{\xi} \delta_{ij}, \quad \Gamma_{00}^0 = -\frac{\delta+1}{\xi}.$$

In other words,

$$\begin{aligned} \nabla \partial_{z_\alpha^i} &= -\frac{\delta}{\xi} d\xi \otimes \partial_{z_\alpha^i}, \nabla \partial_\xi = -\frac{\delta+1}{\xi} dz_\alpha^i \otimes \partial_{z_\alpha^i} - \frac{\delta+1}{\xi} d\xi \otimes \partial_\xi. \\ \nabla dz_\alpha^i &= -\frac{\delta}{\xi} (d\xi \otimes dz_\alpha^i + dz_\alpha^i \otimes d\xi), \quad \nabla d\xi = -\frac{\delta+1}{\xi} d\xi \otimes d\xi. \end{aligned}$$

So we see that

$$|\nabla_{\omega_0} \partial_{z_\alpha^i}|_{\omega_0} \leq C \sim \frac{|\partial_{z_\alpha^i}|_{\omega_0}}{|\xi|^{-\delta}} \sim \frac{|\partial_{z_\alpha^i}|_{\omega_0}}{r}, \quad |\nabla_{\omega_0} \partial_\xi|_{\omega_0} \leq C |\xi|^{-1} \sim \frac{|\partial_\xi|_{\omega_0}}{|\xi|^{-\delta}} \sim \frac{|\partial_\xi|_{\omega_0}}{r}. \quad (34)$$

$$|\nabla_{\omega_0} dz_\alpha^i|_{\omega_0} \leq C |\xi|^{2\delta} \sim \frac{|dz_\alpha^i|_{\omega_0}}{|\xi|^{-\delta}} \sim \frac{|dz_\alpha^i|_{\omega_0}}{r}, \quad |\nabla_{\omega_0} d\xi|_{\omega_0} \leq C |\xi|^{1+2\delta} \sim \frac{|d\xi|_{\omega_0}}{|\xi|^{-\delta}} \sim \frac{|d\xi|_{\omega_0}}{r}. \quad (35)$$

We conclude this section by recalling the Calabi-Yau cone metric in the case when  $K_D^{-1} = \mu L|_D = \mu N_D$  for  $\mu > 0$  and  $D$  has a Kähler-Einstein metric  $\omega_D = \omega_D^{\text{KE}}$  such that  $\text{Ric}(\omega_D^{\text{KE}}) = \mu \cdot \omega_D^{\text{KE}}$ . In this case, note that the Hermitian metric  $h$  satisfies  $\sqrt{-1} \partial \bar{\partial} \log h = \omega_D^{\text{KE}}$ . To find the Calabi-Yau cone metric, it's straightforward to calculate that:

$$\text{Ric}(\omega_0) = -\sqrt{-1} \partial \bar{\partial} \log \omega_0^n = (-n\delta + \mu) \pi_L^* \omega_D^{\text{KE}},$$

where  $n = \dim D + 1$ . So we get the exponent for the Calabi-Yau cone metric:

$$-K_D = \mu N_D \implies \delta = \frac{\mu}{\dim D + 1}. \quad (36)$$

### 3.2 Asymptotical rates of Tian-Yau's Examples

Assume that  $X$  is a Fano manifold of dimension  $n$  and  $D$  is a smooth divisor such that  $-K_X \sim \alpha D$  with  $\mathbb{Q} \ni \alpha > 1$ . By adjunction formula, we get  $-K_D = -K_X|_D - [D] = (\alpha - 1)[D] = (1 - \alpha^{-1})K_X^{-1}$  is still ample, and so  $D$  is also a Fano manifold. Assuming that  $D$  has a Kähler-Einstein metric, Tian-Yau [29] constructed an Asymptotical Conical (AC) Calabi-Yau Kähler metric  $\omega_{\text{TY}}$  on  $X \setminus D$ . The tangent cone at infinity is the conical Calabi-Yau metric on  $C(D, N_D)$  discussed at the end of last section with the exponent:

$$\delta = \frac{\alpha - 1}{n}.$$

By the work of Conlon-Hein [11] (Theorem 5.3 in the Appendix), to find the convergence rate of  $\omega_{\text{TY}}$  to the  $C(D, N_D)$  at infinity, we would like to construct a diffeomorphism  $F : C(D, N_D) \setminus B_R(\varrho) \rightarrow (X \setminus D) \setminus K$  such that

$$\|\nabla_{g_0}^j (F^*(\Omega) - \Omega_0)\|_{g_0} \leq C r^{-\lambda-j}.$$

We will use the diffeomorphism constructed in Section 2.1. Now assume  $D$  is  $(k-1)$ -comfortably embedded. By Theorem 5.5, there exist coordinate charts such that:

$$\begin{cases} z_\beta^1 &= a_{\beta\alpha}(z_\alpha'') z_\alpha^1 + R_{k+1}^1, \\ z_\beta^p &= \phi_{\beta\alpha}^p(z_\alpha'') + R_k^p, \end{cases} \quad \text{for } p = 2, \dots, n.$$

The vector field  $\mathbb{V}$  in (12) becomes:

$$\mathbb{V} = \sum_\alpha \rho_\alpha [\partial_t(t^k \tilde{R}_{k+1}^1)] \otimes \frac{\partial}{\partial w_\beta^1} + \sum_{p=2}^n \sum_\alpha \rho_\alpha [\partial_t(t^k \tilde{R}_k^p)] \otimes \frac{\partial}{\partial w_\beta^p} + \left( \frac{\partial}{\partial t} \right)_\beta.$$

On the total space of  $\tilde{\mathcal{X}}$ , the relative holomorphic form with a pole of order  $\alpha$  along  $\mathcal{D}$  can be written locally as:

$$\Omega = f(t, w) \frac{dw_\beta^1 \wedge \cdots \wedge dw_\beta^n}{(w_\beta^1)^\alpha}$$

with  $f(t, w)$  a locally defined nowhere vanishing holomorphic function. We can then calculate:

$$\begin{aligned} \frac{d}{dt}(\sigma(t)^*\Omega) &= \mathcal{L}_V \Omega = di_V \Omega \\ &= d \left( f(t, w) (w_\beta^1)^{-\alpha} \sum_{\alpha} \rho_{\alpha} [\partial_t(t^k \tilde{R}_{k+1}^1)] \right) \wedge dw_\beta^2 \wedge \cdots \wedge dw_\beta^n \\ &\quad + \sum_{p=2}^n (-1)^{p-1} d \left( f(t, w) (w_\beta^1)^{-\alpha} \sum_{\alpha} \rho_{\alpha} [\partial_t(t^k \tilde{R}_k^p)] \right) dw_\beta^1 \wedge \cdots \wedge \widehat{dw_\beta^p} \wedge \cdots. \end{aligned}$$

Note that  $\tilde{R}_{k+1}^1$  and  $\tilde{R}_k^p$  are holomorphic functions. From this and (32) we see that

$$\begin{aligned} \|F^*\Omega - \Omega_0\|_{\omega_0} &= \|F^*\Omega - \Omega_0\|_{\tilde{\omega}_0} |w_\beta^1|^{\delta+1+\delta(n-1)} \leq Ct^k |w_\beta^1|^{k-\alpha} |w_\beta^1|^{n\delta+1} \\ &= Ct^k |w_\beta^1|^k = Ct^k r^{-k/\delta}. \end{aligned}$$

Here we have used the value  $\delta = \frac{\alpha-1}{n}$  in (36). When restricted to  $\tilde{\mathcal{X}}_0$ , the coordinate  $w_\beta^1$  coincides with the coordinate  $\xi$  on  $N_D \rightarrow D$  defined in the last subsection. Also in changing to the cone metric, we used  $r = a(z)^{1/2} |\xi|^{-\delta} \sim |\xi|^{-\delta} = |w_\beta^1|^{-\delta}$ . By (34) and (35), we see that:

$$\|\nabla_{\omega_0}^j (F^*\Omega - \Omega_0)\|_{\omega_0} \leq Ct^k \frac{|w_\beta^1|^k}{|\xi|^{-j\delta}} = Ct^k r^{-\frac{k}{\delta}-j} = Ct^k r^{-\frac{nk}{\alpha-1}-j} \text{ for any } j \geq 0.$$

So when  $|t|$  is small, we get the required diffeomorphism in Proposition 1.2.

**Example 3.1.**  $(X, D) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \Delta(\mathbb{P}^1))$ .  $\alpha = 2$ ,  $n = 2$ ,  $\delta = (\alpha - 1)/n = 1/2$ . By Proposition 2.3,  $D$  is 1-comfortably embedded (and 1-linearizable) so that  $k = 2$ . So  $\lambda = \frac{k}{\delta} = 4$ .

**Example 3.2.**  $(X, D) \cong (\mathbb{P}^2, \{Z_0^2 + Z_1^2 + Z_2^2 = 0\})$ .  $\alpha = \frac{3}{2}$ ,  $n = 2$ ,  $\delta = (\alpha - 1)/n = 1/4$ . By Proposition 2.4,  $D$  is 0-comfortably embeded (and 0-linearizable) so that  $k = 1$ . So  $\lambda = \frac{k}{\delta} = 4$ .

**Example 3.3.** Assume  $D^{n-1} \subset \mathbb{P}^{N-1}$  is a smooth complete intersection:

$$D = \bigcap_{i=1}^m \{F_i(Z_1, \dots, Z_N) = 0\} \subset \mathbb{P}^{N-1}.$$

where  $m = N - n$  and  $F_i(Z_1, \dots, Z_N)$  is a (generic) homogeneous polynomial of degree  $d_i > 0$ . Denote the affine cone over  $D$  in  $\mathbb{C}^N$  and projective cone over  $D$  inside  $\mathbb{P}^N$  by

$$C(D, H) = \bigcap_{i=1}^m \{F_i(z_1, \dots, z_N) = 0\} \subset \mathbb{C}^N.$$

$$\overline{C}(D, H) = \bigcap_{i=1}^m \{F_i(Z_1, \dots, Z_N)\} \subset \mathbb{P}^N.$$

Notice that since we have assumed that  $D$  is a complete intersection, it's well known that  $D$  is projectively normal in  $\mathbb{P}^{N-1}$  which implies that its projective cone inside  $\mathbb{P}^N$  is normal and hence coincides with its normalization  $\overline{C}(D, H)$ .

Now assume  $G_i(Z_0, Z_1, \dots, Z_N)$  is a generic homogeneous polynomial of degree  $e_i$  with  $e_i < d_i$  for each  $i = 1, \dots, m$ . In particular  $G_i(1, z_1, \dots, z_N)$  a polynomial of degree  $e_i$ . We construct a degeneration:

$$\mathcal{X} = \bigcap_{i=1}^m \{F_i(Z_1, \dots, Z_N) + (tZ_0)^{d_i - \deg G_i} G_i(tZ_0, Z_1, \dots, Z_N) = 0\} \subset \mathbb{P}^N \times \mathbb{C}.$$

For simplicity, we assume that  $\mathcal{X}_1$  is smooth. This degenerates the variety  $\mathcal{X}_1 \subset \mathbb{P}^N$  to  $\overline{C}(D, H)$ . In fact,  $\mathcal{X}$  is a degeneration of  $\mathcal{X}_1$  generated by the one parameter subgroup of projective transformations:

$$[Z_0, Z_1, \dots, Z_N] \rightarrow [t^{-1}Z_0, Z_1, \dots, Z_N].$$

Away from  $\{Z_0 = 0\}$ , we have the deformation of  $C(D, H)$ :

$$\mathcal{X}^\circ = \bigcap_{i=1}^m \{F_i(z_1, \dots, z_N) + t^{d_i - \deg G_i} G_i(t, z_1, \dots, z_N) = 0\} \subset \mathbb{C}^N \times \mathbb{C}.$$

We claim that the degeneration  $\mathcal{X}$  coincides with the one obtained in the contracted deformation to the normal cone construction as in the introduction. This can be readily seen from the graph construction for the deformation to the normal cone, which was recalled in Section 2.2.2. The coincidence of  $\overline{C}(D, H)$  with the central fibre from the contracted deformation to the normal cone can also be verified directly by using Lemma 2.2 and the projective normality of  $D$ .

By adjunction formula, we know that  $-K_{\mathcal{X}_1} = (N + 1 - \sum_{i=1}^m d_i)H$  and  $-K_D = (N - \sum_{i=1}^m d_i)H$ . Consider the hyperplane section  $D = \mathcal{D}_1 = \mathcal{X}_1 \cap \{Z_0 = 0\} \subset \mathcal{X}_1$ . Then if we assume  $\sum_{i=1}^m d_i \leq N - 1$ , we are in the above Tian-Yau's setting with  $\alpha := N + 1 - \sum_{i=1}^m d_i \geq 2$ .

By Appendix 5.4,  $\mathbf{T}_C^1$  can be calculated as a quotient ring. If we define  $w = -\min_{i=1}^m \{d_i - e_i\}$ , then the classifying map  $\mathbf{I}_{\mathcal{X}^\circ}$  defined in the introduction vanishes to the order  $|w|$ , and we see that the reduced Kodaira-Spencer class satisfies:

$$\mathbf{KS}_{\mathcal{X}^\circ}^{\text{red}} = \frac{1}{|w|!} \frac{d^{|w|}}{dt^{|w|}} \Big|_{t=0} \mathbf{I}_{\mathcal{X}^\circ}(t) = \left[ \left\{ \binom{d_i - e_i}{|w|} t^{d_i - e_i - |w|} G_i(1, z_1, \dots, z_N) \right\}_{i=1}^m \right] \Big|_{t=0} =: [\mathcal{G}] \in \bigoplus_{i=1}^m \mathbf{T}^1(-(d_i - e_i)).$$

So if we assume that the image of  $\mathcal{G}$  in  $\mathbf{T}_C^1$  is nonzero, then the weight of deformation  $w(X, D)$  of  $\mathbf{KS}_{\mathcal{X}^\circ}^{\text{red}}$  is equal to the weight of  $[\mathcal{G}]$ .

Let's assume that  $n \geq 3$  from now on. By Theorem 1.1, we know that the divisor  $D$  is  $(|w| - 1)$ -comfortably embedded into  $X$  (but not  $|w|$ -comfortably embedded into  $X$ ).

**Remark 3.1.** As pointed out by the referee, for this class of examples this may not be surprising since we have explicit expressions:

$$\mathcal{X}_1 = \bigcap_{i=1}^m \{F_i(Z_1, \dots, Z_N) + Z_0^{d_i - e_i} G_i(Z_0, Z_1, \dots, Z_N) = 0\} \subset \mathbb{P}^N.$$

Noting that  $|w| = \min_{i=1}^m \{d_i - e_i\}$ , it's immediate that

$$\mathcal{O}_{\mathcal{X}_1}/\mathcal{I}_D^{|w|} \cong \mathcal{O}_{\mathcal{X}_0}/\mathcal{I}_D^{|w|},$$

using the fact that  $\mathcal{I}_D(U_{\{Z_i \neq 0\} \cap \mathcal{X}_1}) = \left( \left\langle \frac{Z_0}{Z_i} \right\rangle + \mathcal{I}_{\mathcal{X}_1} \right) / \mathcal{I}_{\mathcal{X}_1}$ . In other words,  $(\mathcal{X}_1, D)$  is  $(|w| - 1)$ -linearizable. Then by Remark 5.4, when  $n \geq 3$ , we know that  $D$  is  $(|w| - 1)$ -comfortably embedded. So we get  $m(X, D) \geq |w|$ . Of course, the conclusion in Theorem 1.1 is stronger, saying that this is an equality for the more general case without using such explicit defining equations.

So by the above calculation, we see that the asymptotic rate of holomorphic form is given by

$$\lambda = \frac{|w|}{\delta} = \frac{n|w|}{\alpha - 1}.$$

If furthermore  $e_i \leq d_i - 2$ , then

$$\lambda = \frac{|w|}{\delta} = \frac{n|w|}{\alpha - 1} = \frac{n \cdot \min_{i=1}^m \{d_i - e_i\}}{N - \sum_{i=1}^m d_i}.$$

In this way, we can give an algebraic interpretation of the corresponding calculations in [11].

1. ([11, Example 1]). Smoothing of the cubic cone:

$$C = \left\{ z \in \mathbb{C}^4; \sum_{i=1}^4 z_i^3 = 0 \right\} \rightsquigarrow M = \left\{ z \in \mathbb{C}^4; \sum_{i=1}^4 z_i^3 = \sum_{i,j} a_{ij} z_i z_j + \sum_k a_k z_k + \epsilon \right\}.$$

where  $a_{ij}, a_i, \epsilon$  are small (generic) constants. We have

$$\mathbf{T}_C^1 = \frac{\mathbb{C}[z_1, \dots, z_4]}{\langle z_1^2, \dots, z_4^2 \rangle} = \bigoplus_{\nu=-3}^1 \mathbf{T}_C^1(\nu).$$

With the earlier notation,  $G(Z_0, \dots, Z_4) = \sum_{i,j} a_{ij} Z_i Z_j + \sum_k a_k Z_k Z_0 + \epsilon Z_0^2$  with

$$[\mathcal{G}] = \left[ \sum_{ij} a_{ij} z_i z_j + \sum_k a_k z_k + \epsilon \right] \in \mathbf{T}_C^1(-1) + \mathbf{T}_C^1(-2) + \mathbf{T}_C^1(-3).$$

Note that we assume  $a_{ij}$  are generic if it's not zero. So we get

$$\lambda = \begin{cases} \frac{3 \cdot 3}{4-3} = 9, & \text{if } a_{ij} = a_k = 0 \\ \frac{3 \cdot 2}{4-3} = 6, & \text{if } a_{ij} = 0, a_k \neq 0 \\ \frac{3 \cdot 1}{4-3} = 3, & \text{if } a_{ij} \neq 0. \end{cases}$$

2. ([11, Example 2]). Smoothing of the complete intersection:

$$C = \left\{ z \in \mathbb{C}^5; f_1 = \sum_{i=1}^5 z_i^2 = 0, f_2 = \sum_{i=1}^5 \lambda_i z_i^2 = 0 \right\} \rightsquigarrow M = \{z \in \mathbb{C}^5; f_1(z) = f_2(z) = \epsilon\}.$$

Here  $\lambda_i$  are distinct complex numbers. We have:

$$\mathbf{T}_C^1 = \frac{\mathbb{C}[z_1, \dots, z_5]^{\oplus 2}}{\text{Im} \begin{pmatrix} z_1 & \dots & z_5 \\ \lambda_1 z_1 & \dots & \lambda_5 z_5 \end{pmatrix}} = \mathbf{T}_C^1(-2).$$

Because the images of  $\mathcal{G} = (-\epsilon, -\epsilon)$  is not zero inside  $\mathbf{T}_C^1$ , we have  $\lambda = \frac{3 \cdot 2}{5-2-2} = 6$ .

3. ([11, Example 3]). Smoothing of the ordinary double point:

$$C = \left\{ z \in \mathbb{C}^{n+1}; \sum_{i=1}^{n+1} z_i^2 = 0 \right\} \rightsquigarrow M = \left\{ z \in \mathbb{C}^{n+1}; \sum_{i=1}^{n+1} z_i^2 = \sum_{i=1}^{n+1} a_i z_i + \epsilon \right\}.$$

$$\mathbf{T}_C^1 = \frac{\mathbb{C}[z_1, \dots, z_{n+1}]}{\langle z_1, \dots, z_{n+1} \rangle} = \mathbf{T}_C^1(-2).$$

$G(Z_0, \dots, Z_{n+1}) = \sum_{i=1}^{n+1} a_i Z_i + \epsilon Z_0$ . So  $[G(1, z_1, \dots, z_n)] = [\sum_{i=1}^{n+1} a_i z_i + \epsilon] = [\epsilon]$  is of weight  $-2$ . So we have  $\lambda = \frac{n \cdot 2}{n+1-2} = \frac{2n}{n-1}$ .

**Remark 3.2.** If  $n = 2$ , then  $D \hookrightarrow X$  is isomorphic to  $\Delta(\mathbb{P}^1) \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  where  $\Delta : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is the diagonal embedding which was studied in Section 2.3. The identification is easily constructed:

$$\begin{aligned} (\mathbb{P}^1 \times \mathbb{P}^1, \Delta(\mathbb{P}^1)) &\longrightarrow (X, D) = (\{Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 = 0\}, \{Z_0 = 0\} \cap X) \\ ([X_0, X_1], [Y_0, Y_1]) &\mapsto [X_0 Y_1 - X_1 Y_0, \sqrt{-1}(X_0 Y_1 + X_1 Y_0), (X_0 Y_0 + X_1 Y_1), \sqrt{-1}(X_0 Y_0 - X_1 Y_1)]. \end{aligned}$$

## 4 Analytic compactification

### 4.1 Sketch of the proof

As before, denote  $U = L \setminus D$ . Denote the standard complex structure on  $U$  by  $J_0$ . Assume that we have a complex structure  $J$  on some neighborhood  $U_\epsilon$  of  $D$ . Denote  $\Phi = J - J_0$ . We assume the index  $v \in \{1, \bar{1}\}$  associates to the fiber variable  $\xi = z_\alpha^1$ ,  $h \in \{2, \dots, n, \bar{2}, \dots, \bar{n}\}$  associates to the base variables  $\{z_\alpha^2, \dots, z_\alpha^n\}$ . By abuse of notations, we decompose  $\Phi$  into four types of components:

$$\Phi = \Phi_v^h + \Phi_v^v + \Phi_h^v + \Phi_h^h = \phi_v^h dz^v \otimes \partial_{z^h} + \phi_v^v dz^v \otimes \partial_{z^v} + \phi_h^v dz^v \otimes \partial_{z^v} + \phi_h^h dz^h \otimes \partial_{z^h}.$$

We assume  $\Phi$  satisfies  $|\nabla^j \Phi|_{\omega_0} \leq C|r|^{-\lambda-j} \sim |\xi|^{\delta(\lambda+j)}$ . We first need to transform this estimate to the corresponding estimate with respect to  $\tilde{\omega}_0$ . For this, note that we know the basic tensors satisfy (34) and (35). So we can equivalently assume  $\Phi$  satisfies:

$$|(\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \Phi) \otimes (dz^v)^{\otimes j_1} \otimes (dz^h)^{\otimes j_2}|_{\omega_0} \leq C|r|^{-\lambda-j} = C|\xi|^{\delta(\lambda+j)}. \quad (37)$$

Recall the norm in Section 3.1:

$$|dz^v|_{\omega_0} \leq C|\xi|^{\delta+1}, |dz^h|_{\omega_0} \leq C|\xi|^\delta \implies |(dz^v)^{\otimes j_1} \otimes (dz^h)^{\otimes j_2}|_{\omega_0} \leq |\xi|^{j_1(\delta+1)+j_2\delta} = |\xi|^{\delta j + j_1}.$$

Also we have:

$$|dz^v \otimes \partial_{z^h}|_{\omega_0} \leq |\xi|, |dz^h \otimes \partial_{z^v}|_{\omega_0} \leq |\xi|^{-1}, |dz^v \otimes \partial_{z^v}|_{\omega_0} \leq C, |dz^h \otimes \partial_{z^h}|_{\omega_0} \leq C.$$

By these inequalities, it's easy to see that:

$$|\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \phi_v^h| \leq |\xi|^{\lambda\delta-1-j_1}, |\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \phi_h^v| \leq |\xi|^{\lambda\delta+1-j_1}, |\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \phi_v^v| \leq |\xi|^{\lambda\delta-j_1}, |\partial_{z^v}^{j_1} \partial_{z^h}^{j_2} \phi_h^h| \leq |\xi|^{\lambda\delta-j_1}. \quad (38)$$

The  $(0, 1)$  vector under the new complex structure  $J$  is given by

$$\frac{1}{2}(1 + \sqrt{-1}J) \frac{\partial}{\partial \bar{z}^i} = \frac{\partial}{\partial \bar{z}^i} + \frac{\sqrt{-1}}{2} \phi_i^{\bar{j}} \frac{\partial}{\partial \bar{z}^j} + \frac{\sqrt{-1}}{2} \phi_i^k \frac{\partial}{\partial z^k}.$$

Denote  $\eta = \lambda\delta$  and  $\rho = |\xi| = |z^1|$ . Then from (38), we can write:

$$\begin{pmatrix} \phi_i^{\bar{j}} \end{pmatrix} \sim \begin{pmatrix} O(\rho^\eta)_{1 \times 1} & O(\rho^{\eta+1})_{1 \times (n-1)} \\ O(\rho^{\eta-1})_{(n-1) \times 1} & O(\rho^\eta)_{(n-1) \times (n-1)} \end{pmatrix} \sim \begin{pmatrix} \phi_i^k \end{pmatrix}. \quad (39)$$

**Lemma 4.1.** *When  $\rho$  is sufficiently small, the matrix  $\left(\delta_i^{\bar{j}} + \frac{\sqrt{-1}}{2} \phi_i^{\bar{j}}\right)$  is invertible. We have:*

$$\begin{pmatrix} a_i^{\bar{k}} \end{pmatrix} := - \left( \delta_i^{\bar{j}} + \frac{\sqrt{-1}}{2} \phi_i^{\bar{j}} \right)^{-1} \begin{pmatrix} \frac{\sqrt{-1}}{2} \phi_j^{\bar{k}} \end{pmatrix} \sim \begin{pmatrix} O(\rho^\eta)_{1 \times 1} & O(\rho^{\eta+1})_{1 \times (n-1)} \\ O(\rho^{\eta-1})_{(n-1) \times 1} & O(\rho^\eta)_{(n-1) \times (n-1)} \end{pmatrix}.$$

*Proof.* First we can eliminate the lower left part:

$$\begin{pmatrix} 1 & 0 \\ -(1 + \frac{\sqrt{-1}}{2} \phi_1^{\bar{1}})^{-1} \left( \frac{\sqrt{-1}}{2} \phi_1^{\bar{i} > 1} \right) & I_{(n-1) \times (n-1)} \end{pmatrix} \begin{pmatrix} 1 + \frac{\sqrt{-1}}{2} \phi_1^{\bar{1}} & \frac{\sqrt{-1}}{2} \phi_{k > 1}^{\bar{1}} \\ \frac{\sqrt{-1}}{2} \phi_1^{\bar{j} > 1} & (\delta_k^{\bar{j}} + \frac{\sqrt{-1}}{2} \phi_k^{\bar{j}})_{(n-1) \times (n-1)} \end{pmatrix} \\ \sim \begin{pmatrix} 1 + O(\rho^\eta) & O(\rho^{\eta+1})_{1 \times (n-1)} \\ 0 & I_{(n-1) \otimes (n-1)} + (O(\rho^\eta) + O(\rho^{2\eta}))_{(n-1) \times (n-1)} \end{pmatrix}.$$

This clearly implies:

$$\left( I + \frac{\sqrt{-1}}{2} \phi_i^{\bar{j}} \right)^{-1} \sim \begin{pmatrix} O(1) & O(\rho^{\eta+1})_{1 \times (n-1)} \\ O(\rho^{\eta-1})_{(n-1) \times 1} & O(1)_{(n-1) \times (n-1)} \end{pmatrix}$$

Multiplying this by  $(\phi_i^{\bar{k}})$  in (39), we get the lemma.  $\square$

To get an analytic compactification of the complex structure  $J$ , we want to solve for a map  $z : \mathbb{D}_R^n \rightarrow \mathbb{D}_{2R}^n \subset \mathbb{C}^n$  where  $\mathbb{D}_R^n = \{(\zeta^1, \dots, \zeta^n) \in \mathbb{C}^n; |\zeta^j| \leq R\}$ , such that  $z$  is a homeomorphism onto the image and is holomorphic with respect to  $J_0$  and  $J$ . For the map  $z$  to be holomorphicity,  $dz(\partial/\partial \bar{\zeta}^l)$  should be a  $(0, 1)$ -vector for any  $l \geq 1$ . It's easy to see that  $z^i = z^i(\zeta)$  must solve the following equations:

$$\frac{\partial z^i}{\partial \bar{\zeta}^l} + \sum_{p=1}^n a_p^i(z) \frac{\partial \bar{z}^p}{\partial \bar{\zeta}^l} = 0, \quad i, l = 1, \dots, n. \quad (40)$$

We can write these out into components: ( $1 < j, k, m \leq n$ )

$$\begin{cases} \frac{\partial z^1}{\partial \bar{\zeta}^1} + (a_1^1 \sim \rho^\eta) \frac{\partial \bar{z}^1}{\partial \bar{\zeta}^1} + (a_m^1 \sim \rho^{\eta+1}) \frac{\partial \bar{z}^m}{\partial \bar{\zeta}^1} = 0 \\ \frac{\partial z^1}{\partial \bar{\zeta}^k} + (a_1^1 \sim \rho^\eta) \frac{\partial \bar{z}^1}{\partial \bar{\zeta}^k} + (a_m^1 \sim \rho^{\eta+1}) \frac{\partial \bar{z}^m}{\partial \bar{\zeta}^k} = 0 \\ \frac{\partial z^j}{\partial \bar{\zeta}^1} + (a_1^j \sim \rho^{\eta-1}) \frac{\partial \bar{z}^1}{\partial \bar{\zeta}^1} + (a_m^j \sim \rho^\eta) \frac{\partial \bar{z}^m}{\partial \bar{\zeta}^1} = 0 \\ \frac{\partial z^j}{\partial \bar{\zeta}^k} + (a_1^j \sim \rho^{\eta-1}) \frac{\partial \bar{z}^1}{\partial \bar{\zeta}^k} + (a_m^j \sim \rho^\eta) \frac{\partial \bar{z}^m}{\partial \bar{\zeta}^k} = 0. \end{cases} \quad (41)$$

**Remark 4.1.** *The existence of complex analytic coordinate system for any integrable almost complex structures  $J$  is a classical result in complex geometry. If the complex structure is analytic, the existence follows from the Frobenius theorem. When  $J$  is  $C^{2n+\alpha}$  this was the celebrated Newlander-Nirenberg theorem [22]. Nijenhuis-Woolf [23] proved the existence when  $J$  is only  $C^{1+\alpha}$ . Later Malgrange [21] gave a short proof of the  $C^{1+\alpha}$ -case by using some gauge fixing to reduce the existence to the analytic case. More recently, Hill-Taylor [19] generalized Malgrange's method to deal with the case when  $J$  is only  $C^\alpha$  which seems to be the weakest assumption on the regularity of complex structures in the literature.*

By the above remark, if we assume  $\eta > 1$ , then the existence of solutions to the system (41) follows from the work of [19]. We want also to deal with the case when we only assume  $\eta > 0$  when the component  $a_{\bar{1}}^j$  might blows up if  $\eta < 1$ . Since  $J$  is assumed to be smooth outside  $D$ , this can also be seen as some removable singularity problem. We will solve the system (41) following the work of Newlander-Nirenberg [22]. One should also be able to adapt the work of Nijenhuis-Woolf [23], Malgrange [21] to the current setting to prove the compactification (extension) of the complex structures considered here. See also Remark 4.2.

We first recall the important homotopy operator in [22]. For a vector of  $n$  complex-valued functions  $F = (f_{\bar{1}}, \dots, f_{\bar{n}})$ , denote ([22, (2.5)]):

$$\mathbb{T}F = \sum_{s=0}^{n-1} \frac{(-1)^s}{(s+1)!} \sum' T^{j_1} \bar{\partial}_{j_1} \dots T^{j_s} \bar{\partial}_{j_s} \cdot T^k f_{\bar{k}}.$$

where  $\sum'$  denote the summation over all  $(s+1)$ -tuples with  $j_1, \dots, j_s, k$  distinct, and

$$\begin{aligned} T^1 f(\zeta) &= \frac{1}{2\pi i} \iint_{0 < |\tau| < R} \frac{f(\tau, \zeta^2, \dots, \zeta^n)}{\zeta^1 - \tau} d\bar{\tau} d\tau, \\ T^j f(\zeta) &= \frac{1}{2\pi i} \iint_{|\tau| < R} \frac{f(\zeta^1, \dots, \zeta^{j-1}, \tau, \zeta^j, \dots, \zeta^n)}{\zeta^j - \tau} d\bar{\tau} d\tau, \text{ for } j \geq 2. \end{aligned}$$

For fit our setting, we need to modify this by defining:

$$\tilde{T}^1 f(\zeta) = T^1 f(\zeta^1, \zeta^2, \dots, \zeta^n) - T^1 f(0, \zeta^2, \dots, \zeta^n), \quad \tilde{T}^j f(\zeta) = T^j f(\zeta), \text{ if } j \geq 2.$$

Then by Lemma 4.3 and Lemma 4.4, these operators are well defined for functions  $f$  such that  $f \sim O(|\zeta^1|^{\eta-1})$  and satisfy (see [10, (18)]) the following identities on  $\mathbb{D}_R^* \times \mathbb{D}_R^{n-1}$ :

$$\bar{\partial}_j \tilde{T}^j f = f, j = 1, \dots, n; \quad \text{and} \quad \bar{\partial}_j \tilde{T}^k f = \tilde{T}^k \bar{\partial}_j f, \text{ for } j \neq k. \quad (42)$$

Then we define

$$\tilde{\mathbb{T}}F(\zeta) = \sum_{s=0}^{n-1} \frac{(-1)^s}{(s+1)!} \sum' \tilde{T}^{j_1} \bar{\partial}_{j_1} \dots \tilde{T}^{j_s} \bar{\partial}_{j_s} \cdot \tilde{T}^k f_{\bar{k}}.$$

Then using relation (42) to manipulate, we can easily get the following formula which is a variation of the formula in cf. [22, 2.6] by replacing the operator  $T^j$  by  $\tilde{T}^j$ .

$$\bar{\partial}_j \tilde{\mathbb{T}}F - f_{\bar{j}} = \sum_{s=0}^{n-2} \frac{(-1)^s}{(s+2)!} \sum^j \tilde{T}^{j_1} \bar{\partial}_{j_1} \dots \tilde{T}^{j_s} \bar{\partial}_{j_s} \cdot \tilde{T}^k (\bar{\partial}_j f_{\bar{k}} - \bar{\partial}_k f_{\bar{j}}). \quad (43)$$

where  $\sum^j$  denotes the summation over all  $(s+1)$ -tuples with  $j_1, \dots, j_s, k$  distinct and different from  $j$ . From (40), we will denote

$$f_{\bar{l}}^i = - \sum_{p=1}^n a_{\bar{p}}^i(z) \frac{\partial \bar{z}^p}{\partial \zeta^i}, \quad F^i = (f_{\bar{1}}^i, f_{\bar{2}}^i, \dots, f_{\bar{n}}^i) = \sum_{l=1}^n f_{\bar{l}}^i d\zeta^{\bar{l}}. \quad (44)$$

Denote also  $\mathfrak{z}^i(\zeta) = z^i(\zeta) - \zeta^i$ . We then want to transform equations (40) into:

$$z^i = \zeta^i + \tilde{\mathbb{T}}(F^i(z)) \iff \mathfrak{z}^i = \tilde{\mathbb{T}}(F^i(\zeta + \mathfrak{z})) \iff \mathfrak{z} = \mathfrak{J}[\mathfrak{z}]. \quad (45)$$

We will show in Lemma 4.9 that the solution to this equation with the appropriate control is indeed the solution to (40). To get compatible solution to the system (41), we prescribe asymptotically behaviors:

$$z^1 = \zeta^1 + O(\rho^{1+\eta}), \quad z^j = \zeta^j + O(\rho^\eta) \iff \mathfrak{z}^1 \sim O(\rho^{1+\eta}), \quad \mathfrak{z}^k \sim O(\rho^\eta). \quad (46)$$

Here and in the following, we still denote  $\rho = |\zeta^1|$  since  $|\zeta^1|$  and  $|z^1|$  is comparable with this prescription. If we denote  $h$  the index  $\{2, \dots, n\}$ , then the precise meaning of (46) is the following

$$\left| \partial_{\zeta^1}^{l_1} \partial_{\zeta^h}^{l_2} (z^1 - \zeta^1) \right| \leq C(l_1, l_2) |\zeta^1|^{1+\eta-l_1}, \quad \left| \partial_{\zeta^1}^{l_1} \partial_{\zeta^h}^{l_2} (z^h - \zeta^h) \right| \leq C(l_1, l_2) |\zeta^1|^{\eta-l_1}, \text{ for all } l_1, l_2 \geq 0.$$



Under this prescription, by using (44) and the asymptotic behavior of  $a_{\bar{p}}^i$ , we can show (Lemma 4.8) that

$$\begin{aligned} (f_{\bar{1}}^1, f_{\bar{m}}^1) &\sim (O(\rho^\eta + \rho^{2\eta}), O(\rho^{2\eta+1} + \rho^{\eta+1}) \sim (\rho^\eta, \rho^{\eta+1}), \\ (f_{\bar{1}}^j, f_{\bar{m}}^j) &\sim (O(\rho^{\eta-1} + \rho^{2\eta-1}), O(\rho^{2\eta} + \rho^\eta)) \sim (\rho^{\eta-1}, \rho^\eta). \end{aligned}$$

Then we can show that (Lemma 4.6):

$$\tilde{\mathbb{T}}[F^1] \sim O(\rho^{\eta+1}), \quad \tilde{\mathbb{T}}[F^k] \sim O(\rho^\eta) \text{ for } k \geq 2.$$

This is compatible with the prescription in (46) and should allow us to use the arguments in [22] to solve the system (45). However, to use the contraction-iteration principle, we have to relax asymptotic behaviors in (46) a little bit by replacing  $\eta$  by any  $\nu$  such that  $0 < \nu < \eta$  (See Lemma 4.7). This might seem a loss. But actually later we will gain this  $\epsilon$  back using the analyticity of transition functions.

More precisely, in the next subsection, we will introduce weighted multiple Hölder norm  $\|\cdot\|_{n+n\alpha,(\nu+1,\nu)}$  and show in Theorem 4.1 that, for any  $\mathfrak{z}, \tilde{\mathfrak{z}}$  satisfying that when  $R$  is sufficiently small and  $\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)} \leq 1$ ,  $\|\tilde{\mathfrak{z}}\|_{n+n\alpha,(\nu+1,\nu)} \leq 1$ , then the following estimates hold:

1.

$$\|\mathfrak{J}[\mathfrak{z}]\|_{n+n\alpha,(\nu+1,\nu)} \leq 1. \quad (47)$$

2.

$$\|\mathfrak{J}[\mathfrak{z}] - \mathfrak{J}[\tilde{\mathfrak{z}}]\|_{n+n\alpha,(\nu+1,\nu)} \leq \frac{1}{2} \|\mathfrak{z} - \tilde{\mathfrak{z}}\|_{n+n\alpha,(\nu+1,\nu)}. \quad (48)$$

By standard iteration, there is a unique solution to the system (45) such that:

$$\mathfrak{z}^1 \sim O(\rho^{1+\nu}), \quad \mathfrak{z}^j \sim O(\rho^\nu), \text{ or equivalently } z^1 = \zeta^1 + O(\rho^{1+\nu}), \quad z^j = \zeta^j + O(\rho^\nu). \quad (49)$$

In the following  $\mathbb{D}_R = \{\zeta \in \mathbb{C}; |\zeta| \leq R\}$  denotes the closed disc of radius  $R$  with center 0, and  $\mathbb{D}_R^* = \{\zeta \in \mathbb{C}; 0 < |\zeta| \leq R\}$  denotes the punctured closed disc. We need to show that the map  $\zeta \mapsto z$  gives a coordinate chart for  $\zeta \in \mathbb{D}_R^*$  when  $R$  is sufficiently small. First note that  $\{z^i(\zeta)\}$  is identity for  $\zeta^1 = 0$  and is Hölder continuous on  $\{\zeta^1 = 0\}$ . Secondly on  $\mathbb{U}_R = \mathbb{D}_R^* \times \mathbb{D}_R^{n-1}$ , consider the Jacobian

$$\mathbb{J} = \begin{pmatrix} \frac{\partial(z^i, \bar{z}^i)}{\partial(\zeta^j, \bar{\zeta}^j)} \end{pmatrix}.$$

By the similar argument as that in the proof Lemma 39, it's easy to see that  $\mathbb{J}$  is invertible if  $R$  is very small. So on  $\mathbb{U}_R$ ,  $\zeta \mapsto z$  is a local diffeomorphism to its image. We just need to show that it's an injective map and hence a homeomorphism.

To do this, we decompose the coordinate change in (46) into two steps. First we let

$$y^1 = z^1(\zeta) = \zeta^1 + O(|\zeta^1|^{1+\nu}), \quad y^k = \zeta^k \text{ for } k \geq 2. \quad (50)$$

Since the Jacobian matrix is invertible and  $C^\nu$ , the map is a  $C^{1,\nu}$ -diffeomorphism and is clearly a change of coordinates. We can express  $\zeta$  in terms of  $y$  to get:

$$\zeta^1 = y^1 + O(|y^1|^{1+\nu}), \quad \zeta^k = y^k \text{ for } k \geq 2.$$

Now we can write the map in (46) as:

$$z^1 = y^1, \quad z^k = y^k + O(|y^1|^\nu) \text{ for } k \geq 2.$$

We just need to show this is injective. We assume  $z(y) = z(\tilde{y})$ . Then  $y^1 = \tilde{y}^1$ , and  $z^j(y) = z^j(\tilde{y})$ . On the slice  $y^1 = \tilde{y}^1$ , we connect  $y$  and  $\tilde{y}$  by  $y_t = (1-t)y + t\tilde{y}$ , then we have

$$\begin{aligned} 0 = \|z(\tilde{y}) - z(y)\| &= \sum_{j=1}^n \left| \int_0^1 \sum_{k=1}^n (\partial_{y^k} z^j)(y_t) \cdot (\tilde{y}^k - y^k) dt \right| \\ &= \sum_{j=2}^n \left| \int_0^1 \sum_{k=2}^n (\delta_k^j + O(|y^1|^\nu)) (\tilde{y}^k - y^k) dt \right| \\ &\geq C(1 - R^\nu) \|\tilde{y} - y\|. \end{aligned}$$

So if  $R$  is sufficiently small, we indeed have  $\tilde{y} = y$ .

By the similar argument in [22], we can show in the present more technical set-up (see Lemma 4.9) that the  $\{z^i = \zeta^i + \mathfrak{z}^i\}_{i=1}^n$  are indeed solutions to the system (40).

To see the last statement in Theorem 1.2, note that the transition function on the bundle  $N_D \rightarrow D$  in terms of  $\{z_\alpha^i\}$  are standard ones:

$$z_\beta^1 = a_{\beta\alpha}(z'')z_\alpha^1, \quad z_\beta^k = \phi_{\beta\alpha}^k(z'') \text{ for } k \geq 2.$$

By the asymptotical behavior (49) and its inverse, we see that the transition functions in the  $\zeta$ -coordinates have the shape:

$$\zeta_\beta^1 = a_{\beta\alpha}(\zeta'')\zeta_\alpha^1 + O(|\zeta_\alpha^1|^{\nu+1}), \quad \zeta_\beta^k = \phi_{\beta\alpha}^k(\zeta'') + O(|\zeta_\alpha^1|^\nu).$$

We know that  $\zeta_\beta^i$ , for any  $1 \leq i \leq n$ , is a holomorphic function of  $\zeta_\alpha$  outside  $D$ , and from above expressions it's Hölder continuous across  $D = \{\zeta_\alpha^1 = 0\}$ . So we see that  $\zeta_\beta^i$  is holomorphic across  $D$  and hence is a holomorphic function of  $\zeta_\alpha$ . Denote  $m = \lceil \nu \rceil = \lceil \eta \rceil = \lceil \lambda\delta \rceil$  (Recall that  $\eta = \lambda\delta$  and  $\nu = \eta - \epsilon$  for small  $\epsilon$ ). Then the analyticity of holomorphic functions clearly implies that we must have the following improved transition:

$$\zeta_\beta^1 = a_{\beta\alpha}(\zeta'')\zeta_\alpha^1 + R_{m+1}^1, \quad \zeta_\beta^k = \phi_{\beta\alpha}^k(\zeta'') + R_m^k,$$

where  $R_{m+1}^1 \in \mathcal{I}_D^{m+1}$ ,  $R_m^k \in \mathcal{I}_D^m$ , where  $\mathcal{I}_D$  is the ideal sheaf of  $D$  generated by  $\{\zeta_\alpha^1\}$ . By Theorem 5.5 (see also (10)), we see that in the compactification, the divisor  $D$  is indeed  $(m-1)$ -comfortably embedded. In this way, we prove theorem 1.2.

**Remark 4.2.** In [18], the authors proved an analytic compactification result in the asymptotically cylindrical Calabi-Yau case. Their compactification result depends on classifying asymptotical models of the asymptotically cylindrical Calabi-Yau metrics. In the asymptotically conical case, the classification of models at infinity is not clear at present. So here we just concentrate in a case when the model at infinity is known. In this sense, the result obtained here is a counterpart of [18, Theorem 3.1] in our different setting. In the proof of [18, Theorem 3.1], the authors used gauge fixing and used result of Nijenhuis-Woolf [23]. The method used here is technically different and we aim to give a detailed proof by following the fundamental work of Newlander-Nirenberg. In other words, we decide to solve the system (41) altogether to get the coordinate charts. In the next section, we will write down in details the required estimates for  $\bar{\partial}$  equations and iteration processes.

## 4.2 Integral operator on weighted multiple Hölder space

Suppose  $f$  is a complex-valued function defined on  $\mathbb{D}_R^* \times \mathbb{D}_R^{n-1}$ . Denote  $D_j$  either of the differential operators  $\frac{\partial}{\partial \zeta^j}, \frac{\partial}{\partial \bar{\zeta}^j}$ .  $D^k$  will denote a general  $k$ -th order derivative  $D^k = D_{i_1} \dots D_{i_k}$  with  $i_1, \dots, i_k$  distinct (i.e. we consider only “mixed” derivatives).  $D^{k,j} = D_{i_1} \dots D_{i_k}$  (resp.  $D^{k,\{1,j\}}$ ) will denote such a derivative with the  $i_1, \dots, i_k$  distinct and different from  $j$  (resp.  $\{1, j\}$ ). For a fixed positive  $\alpha < 1$ , we denote the difference quotient operators:

$$\delta_1 f = \frac{f(\tilde{\zeta}^1, \zeta^2, \dots, \zeta^n) - f(\zeta^1, \zeta^2, \dots, \zeta^n)}{|\tilde{\zeta}^1 - \zeta^1|^\alpha} \text{ for } 0 < |\zeta^1| \leq R, 0 < |\tilde{\zeta}^1| \leq R, \zeta^1 \neq \tilde{\zeta}^1.$$

$$\delta_i f = \frac{f(\zeta^1, \dots, \tilde{\zeta}^i, \dots, \zeta^n) - f(\zeta^1, \dots, \zeta^i, \dots, \zeta^n)}{|\tilde{\zeta}^i - \zeta^i|^\alpha} \text{ for } i > 1, |\zeta^i| < R, |\tilde{\zeta}^i| < R, \zeta^i \neq \tilde{\zeta}^i.$$

Denote  $\delta^m = \delta_{j_1} \dots \delta_{j_m}$  for  $0 \leq m \leq n$  and  $j_1, \dots, j_m$  distinct;  $\delta^0$  will denote the identity operator;  $\delta^{m,1}$  will denote such a difference quotient with  $j_1, \dots, j_m$  distinct and different from 1. The following is the standard Schauder estimate for the elliptic operator  $\bar{\partial}$  for a single variable.

**Lemma 4.2.** Assume  $\alpha \in (0, 1)$  is fixed. There exists a constant  $c > 0$  such that, if  $w \in C^{1,\alpha}(\mathbb{D}_1(0))$  satisfies  $\frac{\partial w}{\partial \bar{\zeta}} = f$  in  $\mathbb{D}_1$  and if  $f \in C^{0,\alpha}(\mathbb{D}_1(0))$ , then

$$\|w\|_{C^{1,\alpha}(\mathbb{D}_{1/2})} \leq c (\|w\|_{L^\infty(\mathbb{D}_1)} + \|f\|_{C^{0,\alpha}(\mathbb{D}_1)}). \quad (51)$$

*Proof.* Denote operators:

$$Tf(\zeta) = \frac{1}{2\pi i} \iint_{\mathbb{D}_1} \frac{f(\tau)}{\tau - \zeta} d\tau \wedge d\bar{\tau}, \quad Sw(\zeta) = \frac{1}{2\pi i} \int_C \frac{w(\tau)}{\tau - \zeta} d\tau.$$

Then  $w \in C^{1,\alpha}(\mathbb{D}_1)$  satisfies:

$$w = T\partial_{\bar{z}}w + Sw = Tf + Sw.$$

By Chern [10, Main Lemma], we have

$$\|Tf\|_{C^{1,\alpha}(\mathbb{D}_1)} \leq \|f\|_{C^{0,\alpha}(\mathbb{D}_1)}$$

On the other hand, because  $Sw = w - Tf$  is holomorphic, we have:

$$\|Sw\|_{C^{1,\alpha}(\mathbb{D}_{1/2})} \leq \|Sw\|_{L^\infty(\mathbb{D}_1)} \leq \|w\|_{L^\infty(\mathbb{D}_1)} + \|Tf\|_{L^\infty(\mathbb{D}_1)} \leq \|w\|_{L^\infty(\mathbb{D}_1)} + \|f\|_{L^\infty(\mathbb{D}_1)}.$$

□

We need to extend the above Schauder estimate to the weighted Hölder space. We follow [24, Chapter 2] to define the weighted Hölder norm for functions on the punctured disks. Note that this definition of weighted norm is slightly different from the definition used in for example [18] and [11]. Although two norms may be equivalent in some sense, the norm used here following [24] only uses the Hölder norm for  $x$  and  $y$  with comparable distances to the puncture. To the author's understanding, this constraint is well adapted to the rescaling argument. For any  $s > 0$ , denote the annulus  $\{\zeta^1 \in \mathbb{C}; s < |\zeta^1| < 2s\}$  by  $A(s, 2s)$ . First we define the norm on the annulus:

$$\begin{aligned} [w]_{1,\alpha,s} &:= \sup_{A(s,2s)} |w| + s \sup_{A(s,2s)} |D_1 w| + s^\alpha \sup_{x,y \in A(s,2s)} \frac{|w(x) - w(y)|}{|x - y|^\alpha} + \\ &\quad + s^{1+\alpha} \sup_{x,y \in A(s,2s)} \frac{|D_1 w(x) - D_1 w(y)|}{|x - y|^\alpha}. \end{aligned}$$

The following is the scaling invariant weighted Hölder norm for functions on the punctured disk of radius  $R$ :

$$\|w\|_{C_\nu^{1,\alpha}(\mathbb{D}_R(0))} = \sup_{s \in (0, R/2]} s^{-\nu} [w]_{1,\alpha,s},$$

As pointed out in [24, Corollary 2.1], the following Lemma is important for deriving the rescaled Schauder estimate in Lemma 4.4.

**Lemma 4.3.** *Assume  $f \in C_{\nu-1}^{1,\alpha}(\mathbb{D}_R)$ , then we have*

$$\|\rho^{-\nu} \tilde{T}f\|_{L^\infty(\mathbb{D}_R)} \leq C \|\rho^{1-\nu} f\|_{L^\infty(\mathbb{D}_R)}. \quad (52)$$

*Proof.* We can first estimate:

$$\begin{aligned} |\rho^{-\nu} \tilde{T}f| &= |\zeta|^{-\nu} \left| \iint_{\mathbb{D}_R^*} \left( \frac{f(\tau)}{\tau - \zeta} - \frac{f(\tau)}{\tau} \right) dV \right| = |\zeta|^{-\nu} \left| \iint_{\mathbb{D}_R} \frac{f(\tau)\zeta}{(\tau - \zeta)\tau} dV \right| \\ &\leq \|\rho^{1-\nu} f\|_{L^\infty} |\zeta|^{1-\nu} \iint_{\mathbb{D}_R(0)} \frac{dV}{|\tau - \zeta||\tau|^{2-\nu}} \end{aligned}$$

We split the integral into three parts:

$$\iint_{\mathbb{D}_R(0)} = \iint_{\mathbb{D}_{\rho/2}(0)} + \iint_{\mathbb{D}_{\rho/2}(\zeta)} + \iint_{\mathbb{D}_R(0)/(\mathbb{D}_{\rho/2}(0) \cup \mathbb{D}_{\rho/2}(\zeta))} = \mathbf{I} + \mathbf{II} + \mathbf{III}.$$

The inequality (52) follows from the following estimates:

$$\begin{aligned} \mathbf{I} &\leq C \int_0^{\rho/2} \frac{ds}{s^{1-\nu}\rho/2} \leq C\rho^{\nu-1}, \quad \mathbf{II} \leq C \int_0^{\rho/2} \frac{ds}{\rho^{2-\nu}} \leq C\rho^{\nu-1} \\ \mathbf{III} &\leq C \int_{\rho/2}^R \frac{ds}{s^{2-\nu}} \leq C \left( \left( \frac{\rho}{2} \right)^{\nu-1} - R^{\nu-1} \right). \end{aligned}$$

□

**Lemma 4.4** (Potential lemma). *If  $f \in C_{\nu-1}^{0,\alpha}(\mathbb{D}_R)$ , then  $\tilde{T}f \in C_{\nu}^{1,\alpha}(\mathbb{D}_R)$  and satisfies:*

$$\|\tilde{T}f\|_{C_{\nu}^{1,\alpha}(\mathbb{D}_R)} \leq C\|f\|_{C_{\nu-1}^{0,\alpha}(\mathbb{D}_R)}.$$

*Proof.* Let  $F(\zeta) = \tilde{T}f(\zeta)$ . Let  $\rho = |\zeta|$ . By Lemma 4.2, Lemma 4.3 and standard rescaling argument as in [24, Corollary 2.1], we have:

$$\|\tilde{T}f\|_{C_{\nu}^{1,\alpha}(\mathbb{D}_{R/2})} \leq C\|f\|_{C_{\nu-1}^{0,\alpha}(\mathbb{D}_R)}.$$

To get estimate on  $\mathbb{D}_R \setminus \mathbb{D}_{R/2}$ , we use the explicit formula of  $\tilde{T}$ . As in [10, (18), (26)], we have:

$$F_{\bar{\zeta}} = f(\zeta), \quad F_{\zeta} = \frac{1}{2\pi\sqrt{-1}} \iint_{\mathbb{D}_R(0)} \frac{f(\tau) - f(\zeta)}{(\tau - \zeta)^2} d\bar{\tau} d\tau.$$

So that

$$\left| \frac{F_{\zeta}}{|\zeta|^{\nu-1}} \right| \leq \frac{1}{2\pi} \frac{1}{|\zeta|^{\nu-1}} \iint_{\mathbb{D}_R(0)} \frac{|f(\tau) - f(\zeta)|}{|\tau - \zeta|^2} dV(\tau).$$

We can assume  $R/8 \leq |\zeta| \leq R$ . To estimate the integrals, we split it into two parts:

$$\iint_{\mathbb{D}_R(0)} = \iint_{\mathbb{D}_{R/2}(0)} + \iint_{\mathbb{D}_R(0) \setminus \mathbb{D}_{R/2}(0)} = \mathbf{I} + \mathbf{II}.$$

We then estimate:

$$\begin{aligned} \mathbf{I} &\leq C\|\rho^{1-\nu}f\|_{L^\infty(\mathbb{D}_R)} \frac{1}{R^2} \int_0^{R/8} s^{\nu-1} ds \leq CR^{\nu-1}\|\rho^{1-\nu}f\|_{L^\infty(\mathbb{D}_R(0))}. \\ \mathbf{II} &\leq C \iint_{\mathbb{D}_R(0) \setminus \mathbb{D}_{R/2}(0)} \frac{\|f\|_{C_{\nu-1}^{1,\alpha}} |\tau - \zeta|^\alpha R^{-\alpha}}{|\tau - \zeta|^2} dV \leq C\|f\|_{C_{\nu-1}^{1,\alpha}} R^{-\alpha} \int_0^{2R} s^{\alpha-2+1} ds \leq C\|f\|_{C_{\nu-1}^{1,\alpha}}. \end{aligned}$$

So we get  $\|\rho^{1-\nu}D_1\tilde{T}f\|_{L^\infty} \leq \|f\|_{C_{\nu-1}^{1,\alpha}}$ . Similarly, one can prove that:

$$R^\alpha \sup_{x,y \in A(R/8,R)} \frac{|w(x) - w(y)|}{|x - y|^\alpha} + R^{1+\alpha} \sup_{x,y \in A(R/8,R)} \frac{|D_1w(x) - D_1w(y)|}{|x - y|^\alpha} \leq \|f\|_{C_{\nu-1}^{0,\alpha}},$$

with  $w = \tilde{T}(f)$ . In fact, we can prove the inequality as in [23, Section 6.1e], the only difference is that we need to separate the integral over  $\mathbb{D}_{R/2}(0)$  from each estimate since we only have Hölder estimate for  $x$  and  $y$  of comparable lengths.  $\square$

Similarly to [22, (3.1)-(3.3)], we introduce the weighted multiple-Hölder space by incorporating the weighted 1st order Hölder space for  $\zeta^1$  and the usual 1st order Hölder spaces for the other variables. Formally, we define:

1. (Integral part )

$$\|u\|_{n,\nu} = \sum_{k=0}^{n-1} \left( \frac{R^k}{k!} \sup_{\mathbb{D}_R(0)^* \times \mathbb{D}_R(0)^{n-1}} \left( \frac{|D^{k,1}u|}{|\zeta^1|^\nu} \right) + \frac{R^{k+1}}{(k+1)!} \sup_{\mathbb{D}_R(0)^* \times \mathbb{D}_R(0)^{n-1}} \left( \frac{|D_1D^{k,1}u|}{|\zeta^1|^{\nu-1}} \right) \right).$$

2. (Fractional part i.e. difference quotient part):

$$[u]_{n\alpha,\nu} = \sum_{m=1}^{n-1} \left( \frac{R^{m\alpha}}{m!} \sup \left( \frac{|\delta^{m,1}u|}{|\zeta^1|^\nu} \right) + \frac{R^{(m+1)\alpha}}{(m+1)!} \sup_{s \in (0,R/2)} s^{\alpha-\nu} \sup_{\{\zeta^1, \bar{\zeta}^1 \in A(s,2s)\}} |\delta_1 \delta^{m,1}u| \right).$$

3. (0th-order weighted multiple Hölder norm)

$$\|u\|_{n\alpha,\nu} = \tilde{H}_{\alpha,\nu}[u] = \sup \frac{|u|}{|\zeta^1|^\nu} + [u]_{n\alpha,\nu}$$

4. (1st-order weighted multiple Hölder norm)

$$\begin{aligned}\|u\|_{n+n\alpha,\nu} &= \|u\|_{n,\nu} + \sum_{k=0}^{n-1} \left( \frac{R^k}{k!} [D^{k,1}u]_{n\alpha,\nu} + \frac{R^{k+1}}{(k+1)!} [D_1 D^{k,1}u]_{n\alpha,\nu-1} \right) \\ &= \sum_{k=0}^{n-1} \left( \frac{R^k}{k!} \tilde{H}_{\alpha,\nu}[D^{k,1}u] + \frac{R^{k+1}}{(k+1)!} \tilde{H}_{\alpha,\nu-1}[D_1 D^{k,1}u] \right).\end{aligned}$$

5. (Partial 1st-order weighted multiple Hölder norm)

$$\begin{aligned}\|u\|_{n-1+n\alpha,\nu}^1 &= \sum_{k=0}^{n-1} \frac{R^k}{k!} \sup \tilde{H}_{\alpha,\nu}[D^{k,1}u]. \\ \|u\|_{n-1+n\alpha,\nu}^j &= \sum_{k=0}^{n-2} \left( \frac{R^k}{k!} \sup \tilde{H}_{\alpha,\nu}[D^{k,\{1,j\}}u] + \frac{R^{k+1}}{(k+1)!} \tilde{H}_{\alpha,\nu-1}[D_1 D^{k,\{1,j\}}u] \right) \text{ for } j \geq 2.\end{aligned}$$

6. (Anisotropically-weighted norm for vector of functions) Denote  $\mathfrak{z} = (\mathfrak{z}^1(\zeta), \dots, \mathfrak{z}^n(\zeta))$ ,  $F = (f_1, \dots, f_n)$ . Denote:

$$\begin{aligned}\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)} &= \|\mathfrak{z}^1\|_{n+n\alpha,\nu+1} + \sum_{j=2}^n \|\mathfrak{z}^j\|_{n+n\alpha,\nu}. \\ \|F\|_{n-1+n\alpha,(\nu,\nu+1)} &= \|f_1\|_{n-1+n\alpha,\nu}^1 + \sum_{j=2}^n \|f_j\|_{n-1+n\alpha,\nu+1}^j.\end{aligned}$$

Now we come back to solve the system (45) which is equivalent to:

$$\mathfrak{z} = \tilde{\mathbb{T}}(F^i(\zeta + \mathfrak{z})) = \mathfrak{J}[\mathfrak{z}], \text{ where } F^i = (f_i^i) = \left( -\sum_{p=1}^n a_p^i \frac{\partial \bar{z}^p}{\partial \zeta^i} \right). \quad (53)$$

Arguing as in [22], the following lemma is a consequence of Lemma 4.4.

**Lemma 4.5** (cf. [22, Lemma 4.1, Lemma 4.3]). *We have the following estimates:*

$$\begin{aligned}\left\| \tilde{T}^j D_j f \right\|_{n-1+n\alpha,\nu}^l &\leq \|f\|_{n-1+n\alpha,\nu}^l, \quad j, l = 1, \dots, n, j \neq l; \\ \left\| \tilde{T}^1 f \right\|_{n+n\alpha,\nu+1} &\leq c \|f\|_{n-1+n\alpha,\nu}^1; \\ \left\| \tilde{T}^j f \right\|_{n+n\alpha,\nu} &\leq c \|f\|_{n-1+n\alpha,\nu}^j \text{ for } j \geq 2.\end{aligned} \quad (54)$$

Note that, the operator  $\tilde{T}^1$  improves the weight by 1. Packing these estimates for components of  $F^1, F^j$ , the above Lemma implies:

**Lemma 4.6** (cf. [22, Theorem 4.1]).

$$\left\| \tilde{\mathbb{T}}(F^1) \right\|_{n+n\alpha,\nu+1} \leq c \|F^1\|_{n-1+n\alpha,(\nu,\nu+1)}; \quad \left\| \tilde{\mathbb{T}}(F^j) \right\|_{n+n\alpha,\nu} \leq c \|F^j\|_{n-1+n\alpha,(\nu-1,\nu)}, \text{ for } j \geq 2.$$

The following lemma follows from the decay rate of  $(a_j^i)$  in Lemma 4.1 and the definition of norms defined above. It shows the reason to relax the asymptotics by replacing  $\eta$  by  $\nu$ .

**Lemma 4.7** (cf. [22, Lemma 3.1]). *Suppose  $\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)} \leq 1$ , then*

$$\begin{aligned}\|a_1^1(\zeta + \mathfrak{z})\|_{n+n\alpha,\nu} &\leq KR^{\eta-\nu}(1 + \|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}), \quad \|a_k^1(\zeta + \mathfrak{z})\|_{n+n\alpha,\nu+1} \leq KR^{\eta-\nu}(1 + \|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}). \\ \|a_1^k(\zeta + \mathfrak{z})\|_{n+n\alpha,\nu-1} &\leq KR^{\eta-\nu}(1 + \|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}), \quad \|a_k^j(\zeta + \mathfrak{z})\|_{n+n\alpha,\nu} \leq KR^{\eta-\nu}(1 + \|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}).\end{aligned}$$

**Lemma 4.8** (cf. [22, Lemma 5.1]). *If  $\|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)} \leq 1$ . Then*

$$\|F^1\|_{n-1+n\alpha,(\nu,\nu+1)} \leq CR^{\eta-\nu}(1 + \|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}), \quad \|F^1[\mathfrak{z}] - F^1[\tilde{\mathfrak{z}}]\|_{n+n\alpha,(\nu,\nu+1)} \leq CR^{\eta} \|\mathfrak{z} - \tilde{\mathfrak{z}}\|_{n+n\alpha,(\nu+1,\nu)}. \quad (55)$$

For  $j \geq 2$ , we have:

$$\|F^j\|_{n-1+n\alpha,(\nu-1,\nu)} \leq CR^{\eta-\nu}(1 + \|\mathfrak{z}\|_{n+n\alpha,(\nu+1,\nu)}), \quad \|F^j[\mathfrak{z}] - F^j[\tilde{\mathfrak{z}}]\|_{n+n\alpha,(\nu-1,\nu)} \leq CR^{\eta} \|\mathfrak{z} - \tilde{\mathfrak{z}}\|_{n+n\alpha,(\nu+1,\nu)}. \quad (56)$$

*Proof.*  $f_{\bar{1}}^1 \sim O(\rho^\nu + \rho^{2\nu})$ :

$$\begin{aligned} \left\| a_{\bar{1}}^1 \frac{\partial \bar{z}^1}{\partial \bar{\zeta}^1} \right\|_{n-1+n\alpha, \nu}^1 &\leq \|a_{\bar{1}}^1\|_{n-1+n\alpha, \nu}^1 \left( 1 + \left\| \frac{\partial \bar{\mathfrak{z}}^1}{\partial \bar{\zeta}^1} \right\|_{n-1+n\alpha, 0}^1 \right) \\ &\leq KR^{\eta-\nu} (1 + \|\mathfrak{z}\|_{n+n\alpha, (\nu+1, \nu)}) (1 + \|\bar{\mathfrak{z}}^1\|_{n+n\alpha, \nu+1} R^\nu). \end{aligned}$$

$$\begin{aligned} \left\| a_{\bar{1}}^1(\zeta + \mathfrak{z}) \frac{\partial \bar{z}^1}{\partial \bar{\zeta}^1} - a_{\bar{1}}^1(\zeta + \tilde{\mathfrak{z}}) \frac{\partial \bar{z}^1}{\partial \bar{\zeta}^1} \right\|_{n-1+n\alpha, \nu}^1 &\leq \|a_{\bar{1}}^1(\zeta + \mathfrak{z}) - a_{\bar{1}}^1(\zeta + \tilde{\mathfrak{z}})\|_{n-1+n\alpha, \nu}^1 \left\| \frac{\partial \bar{z}^1}{\partial \bar{\zeta}^1} \right\|_{n-1+n\alpha, 0}^1 \\ &\quad + \|a_{\bar{1}}^1(\zeta + \tilde{\mathfrak{z}})\|_{n-1+n\alpha, \nu} \left\| \frac{\partial(\mathfrak{z}^1 - \tilde{\mathfrak{z}}^1)}{\partial \bar{\zeta}^1} \right\|_{n-1+n\alpha, 0}^1 \\ &\leq KR^\eta \|\mathfrak{z} - \tilde{\mathfrak{z}}\|_{n+n\alpha, (\nu+1, \nu)}. \end{aligned}$$

Note that, similar with the method in our proof that  $\zeta \mapsto z$  gives coordinate charts, in the above estimates, we can estimate the difference of  $a_{\bar{1}}^1(z) - a_{\bar{1}}^1(\tilde{z})$  by decomposing into two parts and then uses mean value theorem to get the above estimate:

$$\begin{aligned} a_{\bar{1}}^1(\zeta + \mathfrak{z}) - a_{\bar{1}}^1(\zeta + \tilde{\mathfrak{z}}) &= [a_{\bar{1}}^1(\zeta + \mathfrak{z}) - a_{\bar{1}}^1(\zeta^1 + \tilde{\mathfrak{z}}^1, \zeta'' + \mathfrak{z}'')] + [a_{\bar{1}}^1(\zeta^1 + \tilde{\mathfrak{z}}^1, \zeta'' + \mathfrak{z}'') - a_{\bar{1}}^1(\zeta^1 + \tilde{\mathfrak{z}}^1, \zeta'' + \tilde{\mathfrak{z}}'')] \\ &\sim R^{\eta+\nu} \|\mathfrak{z}^1 - \tilde{\mathfrak{z}}^1\|_{n+n\alpha, \nu+1} + R^{\eta+\nu} \|\mathfrak{z}'' - \tilde{\mathfrak{z}}''\|_{n+\alpha, \nu}. \end{aligned}$$

$$\left\| a_{\bar{m}}^1 \frac{\partial \bar{z}^m}{\partial \bar{\zeta}^1} \right\|_{n-1+n\alpha, \nu}^1 \leq \|a_{\bar{m}}^1\|_{n-1+n\alpha, \nu+1}^1 \left\| \frac{\partial \bar{\mathfrak{z}}^m}{\partial \bar{\zeta}^1} \right\|_{n-1+n\alpha, -1}^1 \leq KR^{\eta-\nu} \|\mathfrak{z}^m\|_{n+n\alpha, \nu} R^\nu.$$

$$\begin{aligned} \left\| a_{\bar{m}}^1(\zeta + \mathfrak{z}) \frac{\partial \bar{z}^m}{\partial \bar{\zeta}^1} - a_{\bar{m}}^1(\zeta + \tilde{\mathfrak{z}}) \frac{\partial \bar{z}^m}{\partial \bar{\zeta}^1} \right\|_{n-1+n\alpha, \nu}^1 &\leq \|a_{\bar{m}}^1(\zeta + \mathfrak{z}) - a_{\bar{m}}^1(\zeta + \tilde{\mathfrak{z}})\|_{n-1+n\alpha, \nu+1}^1 \left\| \frac{\partial \bar{z}^m}{\partial \bar{\zeta}^1} \right\|_{n-1+n\alpha, -1}^1 \\ &\quad + \|a_{\bar{m}}^1(\zeta + \tilde{\mathfrak{z}})\|_{n-1+n\alpha, \nu+1} \left\| \frac{\partial(\mathfrak{z}^m - \tilde{\mathfrak{z}}^m)}{\partial \bar{\zeta}^1} \right\|_{n-1+n\alpha, -1}^1 \\ &\leq KR^\eta \|\mathfrak{z} - \tilde{\mathfrak{z}}\|_{n+n\alpha, (\nu+1, \nu)}. \end{aligned}$$

We used the estimate:

$$\begin{aligned} a_{\bar{m}}^1(\zeta + \mathfrak{z}) - a_{\bar{m}}^1(\zeta + \tilde{\mathfrak{z}}) &= [a_{\bar{m}}^1(\zeta + \mathfrak{z}) - a_{\bar{m}}^1(\zeta^1 + \tilde{\mathfrak{z}}^1, \zeta'' + \mathfrak{z}'')] + [a_{\bar{m}}^1(\zeta^1 + \tilde{\mathfrak{z}}^1, \zeta'' + \mathfrak{z}'') - a_{\bar{m}}^1(\zeta^1 + \tilde{\mathfrak{z}}^1, \zeta'' + \tilde{\mathfrak{z}}'')] \\ &\sim R^{\eta+\nu+1} \|\mathfrak{z}^1 - \tilde{\mathfrak{z}}^1\|_{n+n\alpha, \nu+1} + R^{\eta+\nu+1} \|\mathfrak{z}'' - \tilde{\mathfrak{z}}''\|_{n+\alpha, \nu}. \end{aligned}$$

Similarly, one can verify that  $f_{\bar{m}}^1 \sim O(\rho^{2\nu+1} + \rho^{\nu+1})$ ,  $f_{\bar{1}}^j \sim O(\rho^{\nu-1} + \rho^{2\nu-1})$  and  $f_{\bar{m}}^j \sim O(\rho^{2\nu} + \rho^\nu)$ .  $\square$

Combining Lemma 4.6 and 4.8, we get:

**Theorem 4.1.** *For any  $\mathfrak{z}, \tilde{\mathfrak{z}}$  satisfying  $\|\mathfrak{z}\|_{(\nu+1, \nu)} \leq 1$ ,  $\|\tilde{\mathfrak{z}}\|_{(\nu+1, \nu)} \leq 1$ , we have*

$$\|\mathfrak{J}(\mathfrak{z})\|_{n+n\alpha, (\nu+1, \nu)} \leq cR^{\eta-\nu} (1 + \|\mathfrak{z}\|_{n+n\alpha, (\nu+1, \nu)});$$

$$\|\mathfrak{J}(\tilde{\mathfrak{z}}) - \mathfrak{J}(\mathfrak{z})\|_{n+n\alpha, (\nu+1, \nu)} \leq cR^\eta \|\tilde{\mathfrak{z}} - \mathfrak{z}\|_{n+n\alpha, (\nu+1, \nu)}.$$

So for  $R$  sufficiently small, we indeed get the desired inequalities (47) and (48) to apply the contraction-iteration principle to get a solution to the system (53).

**Lemma 4.9.** *If  $\mathfrak{z}$  is a solution to the system (53), then  $\mathfrak{z}$  is a solution to (40), i.e.*

$$g_j^i = \frac{\partial z^i}{\partial \bar{\zeta}^l} + \sum_{p=1}^n a_{\bar{p}}^i(z) \frac{\partial \bar{z}^p}{\partial \bar{\zeta}^l} = 0, \quad i, l = 1, \dots, n. \quad (57)$$

*Proof.* We follow the argument in [22]. Using the formula (43) and calculating as in [22, (2.11-2.12)] (see also [23, 4.1.2]) we get the following identity

$$g_j^i = \sum_{s=0}^{n-2} \frac{(-1)^s}{(s+2)!} \sum^j \tilde{T}^{j_1} \bar{\partial}_{j_1} \dots \tilde{T}^{j_s} \bar{\partial}_{j_s} \cdot \tilde{T}^k [(\partial_p a_{\bar{m}}^i)(\zeta)(\bar{\partial}_j \bar{z}^m \cdot g_{\bar{k}}^p - \bar{\partial}_k \bar{z}^m \cdot g_{\bar{j}}^p)] \quad (58)$$

where  $\sum^j$  denotes the summation over all  $(s+1)$ -tuples with  $j_1, \dots, j_s, k$  distinct and different from  $j$ . We claim that from (58) the following holds:

$$\|G^1\|_{n-1+n\alpha,(\nu,\nu+1)} + \|G^j\|_{n-1+n\alpha,(\nu-1,\nu)} \leq CR^\eta(\|G^1\|_{n-1+n\alpha,(\nu,\nu+1)} + \|G^j\|_{n-1+n\alpha,(\nu-1,\nu)}). \quad (59)$$

where we denote  $G^i = (g_{\bar{1}}^i, \dots, g_{\bar{n}}^i)$ . Assuming (59) holds, then when  $R$  is sufficiently small, then we have  $G^i = 0$  and so we indeed get the solution to (57). To verify the claim, we need to estimate the term in the bracket:

$$\mathfrak{G}_{\bar{j}\bar{k}}^i := (\partial_p a_{\bar{m}}^i)(\zeta)(\bar{\partial}_j \bar{z}^m \cdot g_{\bar{k}}^p - \bar{\partial}_k \bar{z}^m \cdot g_{\bar{j}}^p).$$

We will estimate it for different cases of indices.

1.  $(i = 1, j = 1)$  In this case  $k \geq 2$  (since  $k \neq j$  in  $\sum^j$ ).
  - (a)  $(p = 1, m = 1)$   $\mathfrak{G}_{\bar{1}\bar{k}}^1 \sim \rho^{\eta-1}(\rho^{0+\nu+1} + \rho^{\nu+1+\nu}) \sim \rho^\eta \rho^\nu$ .
  - (b)  $(p \geq 2, m = 1)$   $\mathfrak{G}_{\bar{1}\bar{k}}^1 \sim \rho^\eta(\rho^{0+\nu} + \rho^{\nu+1+\nu-1}) \sim \rho^\eta \rho^\nu$ .
  - (c)  $(p = 1, m \geq 2)$   $\mathfrak{G}_{\bar{1}\bar{k}}^1 \sim \rho^\eta(\rho^{\nu-1+\nu+1} + \rho^{0+\nu}) \sim \rho^\eta \rho^\nu$ .
  - (d)  $(p \geq 2, m \geq 2)$   $\mathfrak{G}_{\bar{1}\bar{k}}^1 \sim \rho^{\eta+1}(\rho^{\nu-1+\nu} + \rho^{0+\nu-1}) \sim \rho^\eta \rho^\nu$ .
2.  $(i = 1, j \geq 2)$  In this case  $k$  can be 1.
  - (a)  $(k = 1)$ 
    - i.  $(p = 1, m = 1)$   $\mathfrak{G}_{\bar{j}\bar{1}}^1 \sim \rho^{\eta-1}(\rho^{\nu+1+\nu} + \rho^{0+\nu+1}) \sim \rho^\eta \rho^\nu$ .
    - ii.  $(p \geq 2, m = 1)$   $\mathfrak{G}_{\bar{j}\bar{1}}^1 \sim \rho^\eta(\rho^{\nu+1+\nu-1} + \rho^{0+\nu}) \sim \rho^\eta \rho^\nu$ .
    - iii.  $(p = 1, m \geq 2)$   $\mathfrak{G}_{\bar{j}\bar{1}}^1 \sim \rho^\eta(\rho^{0+\nu} + \rho^{\nu-1+\nu+1}) \sim \rho^\eta \rho^\nu$ .
    - iv.  $(p \geq 2, m \geq 2)$   $\mathfrak{G}_{\bar{j}\bar{1}}^1 \sim \rho^{\eta+1}(\rho^{0+\nu-1} + \rho^{\nu-1+\nu}) \sim \rho^\eta \rho^\nu$ .
  - (b)  $(k \geq 2)$ 
    - i.  $(p = 1, m = 1)$   $\mathfrak{G}_{\bar{j}\bar{k}}^1 \sim \rho^{\eta-1}(\rho^{\nu+1+\nu+1} + \rho^{\nu+1+\nu+1}) \sim \rho^{\eta+\nu} \rho^{\nu+1}$ .
    - ii.  $(p \geq 2, m = 1)$   $\mathfrak{G}_{\bar{j}\bar{k}}^1 \sim \rho^\eta(\rho^{\nu+1+\nu} + \rho^{\nu+1+\nu}) \sim \rho^{\eta+\nu} \rho^{\nu+1}$ .
    - iii.  $(p = 1, m \geq 2)$   $\mathfrak{G}_{\bar{j}\bar{k}}^1 \sim \rho^\eta(\rho^{0+\nu+1} + \rho^{0+\nu+1}) \sim \rho^\eta \rho^{\nu+1}$ .
    - iv.  $(p \geq 2, m \geq 2)$   $\mathfrak{G}_{\bar{j}\bar{k}}^1 \sim \rho^{\eta+1}(\rho^{0+\nu} + \rho^{0+\nu}) \sim \rho^\eta \rho^{\nu+1}$ .
3.  $(i \geq 2, j = 1)$  In this case  $k \geq 2$ . From the expression of  $\mathfrak{G}_{\bar{j}\bar{k}}^i$ , we see that the only difference from the case  $i = 1, j = 1$  lies in the term  $\partial_p a_{\bar{m}}^i$ . We just need to decrease each order by 1 to get
$$\mathfrak{G}_{\bar{1}\bar{k}}^i \sim \rho^\eta \rho^{\nu-1}.$$
4.  $(i \geq 2, j \geq 2)$  In this case,  $k$  can be 1. Again, we see that the only difference with the case  $i = 1, j \geq 2$  lies in the term  $\partial_p a_{\bar{m}}^i$ . So we just need to decrease each order by 1 to get
$$\mathfrak{G}_{\bar{j}\bar{1}}^i \sim \rho^\eta \rho^{\nu-1}, \text{ and } \mathfrak{G}_{\bar{j}\bar{k}}^i \sim \rho^\eta \rho^\nu.$$

Now from item 1, we have that:

$$\begin{aligned} \|g_{\bar{1}}^1\|_{n-1+n\alpha,\nu}^1 &\leq C \sum_{k \geq 2} \|\tilde{T}^k \mathfrak{G}_{\bar{1}\bar{k}}^1\|_{n-1+n\alpha,\nu} \\ &\leq CR^\eta(\|G^1\|_{n-1+n\alpha,(\nu,\nu+1)} + \|G^j\|_{n-1+n\alpha,(\nu-1,\nu)}). \end{aligned}$$

From item 2, we have for  $j \geq 2$ ,

$$\begin{aligned} \|g_{\bar{j}}^1\|_{n-1+n\alpha,\nu}^j &\leq C(\|\tilde{T}^1 \mathfrak{G}_{\bar{j}\bar{1}}^1\|_{n-1+n\alpha,\nu+1}^j + \sum_{k \geq 2} \|\tilde{T}^k \mathfrak{G}_{\bar{j}\bar{k}}^1\|_{n-1+n\alpha,\nu+1}^j) \\ &\leq CR^\eta(\|G^1\|_{n-1+n\alpha,(\nu,\nu+1)} + \|G^j\|_{n-1+n\alpha,(\nu-1,\nu)}). \end{aligned}$$

Note that we have used the fact from (4.5) that the operator  $\tilde{T}^1$  improves the weight from  $\nu$  to  $\nu + 1$ . The same argument apply to item 3 and 4 too. So we indeed get the estimate (59).  $\square$



## 5 Appendices

### 5.1 Appendix I: AC Calabi-Yau metric of Tian-Yau

Let  $(M, g)$  be a non-compact complete Riemannian manifold.  $(M, g)$  will be called asymptotically conical (AC) of order  $\eta$  if there exists a metric cone  $(C(Y), \underline{g})$  with the cone metric  $g_0 = dr^2 + r^2 g_Y$  and a diffeomorphism  $\phi_K : C(Y) \setminus B_R(\underline{g}) \rightarrow M \setminus K$  such that

$$\|\nabla_{g_0}^j(\phi_K^*(g_0) - g_\omega)\|_{C^0} \leq Cr^{-\lambda-j} \text{ for } j \geq 0.$$

Here  $K$  is a compact set in  $M$  and  $B_R(\underline{g})$  is the ball of radius  $R$  around the vertex  $\underline{g}$  of the metric cone. Cheeger-Tian [9] proved that a Ricci-flat complete manifold with maximal volume growth and satisfying suitable integral bounds on curvature tensors is indeed a asymptotically conical Ricci-flat manifold. We will be interested in the case when  $g$  is both Kähler and Ricci-flat. If this is the case, we denote by  $\omega_g$ , or simply  $\omega$ , the Kähler form of  $g$  and call  $g$  or  $\omega$  AC Ricci-flat Calabi-Yau metric. There were many beautiful works on this subject. Tian-Yau [29] (see also Bando-Kobayashi [6]) constructed a class of such AC Calabi-Yau manifold. This work is generalized later by van Coevering [31] in a series of papers, and also refined and clarified by Conlon-Hein ([11], [12]) in detail.

A natural question is to determine the optimal order of such AC Calabi-Yau metric. This issue was studied in detail in Cheeger-Tian [9] and in Conlon-Hein ([11], [12]). The fact important to us is that this optimal rate is related to the rate of convergence of complex structures (see [9, Section 7]). More precisely, we would like to determine the largest number  $\lambda_1 > 0$  such that there exists a diffeomorphism  $\phi_K$  as above and the following inequality holds:

$$\|\nabla_{g_0}^j(\phi_K^*J - J_0)\|_{g_0} \leq Cr^{-\lambda_1-j} \text{ for all } j \geq 0.$$

Now assume that  $X$  is a Fano manifold, i.e.  $-K_X$  is an ample line bundle. Assume  $D$  is a smooth divisor such that  $\alpha D \sim K_X^{-1}$  is a smooth divisor with  $\mathbb{Q} \ni \alpha > 1$ . By adjunction formula, we get  $-K_D = -K_X - [D] = (1 - \alpha^{-1})K_X^{-1}$  is still ample, and so  $D$  is also a Fano manifold. We also assume that  $D$  has a Kähler-Einstein metric, then Tian-Yau [29] constructed an AC Calabi-Yau metric on  $M = X \setminus D$ .

**Theorem 5.1** ([29]). *Under the above assumptions,  $X \setminus D$  admits a complete Ricci-flat Kähler metric  $g$  with Euclidean volume growth. Further more, if we denote  $R(g)$  the curvature tensor of  $g$  and by  $\rho(\cdot)$  the distance function on  $X \setminus D$  from some fixed point with respect to  $g$ , then  $R(g)$  decays at the order of exactly  $\rho^{-2}$  with respect to the  $g$ -norm.*

This construction can be viewed as a generalization of the basic example of Eguchi-Hanson metric  $\omega_{EH}$  in which case we have:

$$X = \mathbb{P}^1 \times \mathbb{P}^1, D = \Delta(\mathbb{P}^1) \cong \mathbb{P}^1,$$

where  $\Delta : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is the diagonal embedding. Note that  $M = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta(\mathbb{P}^1)$  is isomorphic to the deformed conifold  $\{z_1^2 + z_2^2 + z_3^2 = 1\} \subset \mathbb{C}^3$  which via the hyperKähler rotation becomes the ALE metric on the crepant resolution of  $\mathbb{C}^2/\mathbb{Z}^2$ .

**Remark 5.1.** *The assumptions for the existence to hold can be weakened to the following items: 1.  $X$  is a Kähler manifold; 2.  $-K_X = \alpha D$  with  $\mathbb{Q} \ni \alpha > 1$ ; 3. Either “almost ample” in the sense of Tian-Yau in [29], or  $N_D = \alpha^{-1}K_X^{-1}|_D$  is ample. 4.  $D$  has Kähler-Einstein metric. For these technical details, see the nice explanation in [12].*

The tangent cone at infinity of this Tian-Yau metric is the conical Calabi-Yau metric on  $C(D, N_D)$  discussed in Section 3.1. Since the normal bundle is an approximation of  $X$  in a small neighborhood of  $D$ , the idea is to prescribe the metric on  $X \setminus D$  to be asymptotically equivalent to  $\omega_0$  near  $D$  and solve the Monge-Ampère equation for solutions with this behavior at infinity. Conlon-Hein [12] refined Tian-Yau’s construction and studied the asymptotical order of the AC Ricci-flat Kähler metrics.

**Theorem 5.2** ([12]). *In each Kähler class of  $X \setminus D$  and for every  $c > 0$ , there exists a unique AC Calabi-Yau metric  $\omega_c$  on  $X \setminus D$  satisfying*

$$\exp^*(\omega_c) - c\omega_0 = O(r^{-\min\{2-\delta, \frac{n}{\alpha-1}\}}) \text{ with } g_0\text{-derivatives, for any } \delta > 0.$$

Here,  $\exp : N_D \rightarrow X$  denotes the restriction to the normal bundle  $N_D$  of  $D$  in  $X$  of the exponential map of any background Kähler metric on  $X$ ,  $g_0$  denotes the pullback to  $N_D \setminus \{0\}$  of Calabi ansatz Ricci-flat Kähler cone metric on  $-K_D$ .

To obtain this, they used the following general existence and regularity result from their first paper [11].

**Theorem 5.3** ([11]). *Let  $M$  be an open complex manifold of complex dimension  $n \geq 3$  such that  $K_M$  is trivial. Let  $\Omega$  be a holomorphic volume form on  $M$  and let  $L$  be Sasaki-Einstein with associated Calabi-Yau cone  $(C, \Omega_0, \omega_0)$  and radius function  $r$ . Suppose that there exists  $\lambda_1 < 0$ , a compact subset  $K \subset M$ , and a diffeomorphism  $F_K : (1, \infty) \times L \rightarrow M \setminus K$  such that*

$$F_K^* \Omega - \Omega_0 = O(r^{-\lambda_1}) \text{ with } g_0 - \text{derivatives,}$$

where  $g_0$  is the Kähler metric associated to  $\omega_0$ . Let  $\mu < 0$  and assume  $\nu := \max\{\lambda_1, \mu\} \notin \{-2n, -2, \nu_0 - 2\}$ , where  $\nu_0 \geq 1$  denotes the smallest growth rate of a pluriharmonic function on  $C$ . Then for every  $c > 0$ , there exists, in each  $\mu$ -almost compactly supported Kähler class, a unique AC Calabi-Yau metric  $\omega_c$  satisfying

$$F^* \omega_c - c\omega_0 = O\left(r^{\max\{-2n, \nu\}}\right) \text{ with } g_0 - \text{derivatives.}$$

**Remark 5.2.** These results are summarized in the estimate (1). Again the important thing for us is that the estimate of asymptotical rate of convergence depends on the construction of some diffeomorphism. For example, to get Theorem 5.2, Conlon-Hein [12] just used smooth exponential map with respect to any smooth Kähler metric on  $X$  as the comparing diffeomorphism  $F_K$ , which can be seen as a first order approximation. In Proposition 1.2 and Corollary 1.1 we get better rate (essentially optimal one) by constructing diffeomorphisms which are more adapted to the embedding of  $D$  inside  $X$ . Also for special examples in [11], Conlon-Hein constructed the diffeomorphisms in a somehow ad hoc way. Proposition 1.2 and Proposition 1.1 together provide a uniform algebraic interpretation and generalization of their constructions.

## 5.2 Appendix II: Neighborhoods of complex submanifold

Recall from the introduction, we have defined:

**Definition 5.1.**  $S$  is  $k$ -linearizable if its  $k$ -th infinitesimal neighbourhood  $S(k) := (S, \mathcal{O}_X/\mathcal{I}_S^{k+1})$  in  $X$  is isomorphic to its  $k$ -th infinitesimal neighbourhood  $S_N(k) := (S, \mathcal{O}_N/\mathcal{I}_S^{k+1})$  in  $N_S$ . Here we identify  $S$  with the zero section  $S_0$  of  $N_S =: N$ .

**Definition 5.2** ([1, Definition 2.1, 2.2]). 1.  $S$  is  $k$ -splitting into  $X$  (for some  $k \geq 1$ ) if the exact sequence

$$0 \longrightarrow \mathcal{I}_S/\mathcal{I}_S^{k+1} \longrightarrow \mathcal{O}_X/\mathcal{I}_S^{k+1} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

splits as sequence of sheaves of rings.

2. A  $k$ -splitting atlas for  $S \subset X$  is an atlas  $\{(V_\alpha, z_\alpha)\}$  of  $X$  adapted to  $S$  (that is,  $V_\alpha \cap S \neq \emptyset$  implies  $V_\alpha \cap S = \{z_\alpha^1 = \dots = z_\alpha^m = 0\}$ ) such that

$$\left. \frac{\partial^k z_\beta^p}{\partial z^{r_1} \dots \partial z_\alpha^{r_k}} \right|_S \equiv 0,$$

for all  $r_1, \dots, r_k = 1, \dots, m$ , all  $p = m+1, \dots, n$ , and all indices  $\alpha, \beta$  such that  $V_\alpha \cap V_\beta \cap S \neq \emptyset$ .

In the following, if  $S$  is  $k$ -splitting, we will fix a lifting:  $\rho_k : \mathcal{O}_S \rightarrow \mathcal{O}_X/\mathcal{I}_S^{k+1}$ . We also denote by  $\phi_{h,k}$  the natural map

$$\phi_{h,k} : \mathcal{O}_X/\mathcal{I}_S^{h+1} \rightarrow \mathcal{O}_X/\mathcal{I}_S^{k+1}, \text{ for } h \geq k. \quad (60)$$

**Proposition 5.1** ([1, Proposition 2.2]). Assume that  $S$  is  $(k-1)$ -splitting in  $X$ ; let  $\rho_{k-1} : \mathcal{O}_S \rightarrow \mathcal{O}_X/\mathcal{I}_S^k$  be a  $(k-1)$ -th order lifting, and  $\mathfrak{V} = \{(V_\alpha, \phi_\alpha)\}$  a  $(k-1)$ -splitting atlas adapted to  $\rho_{k-1}$ . Let  $\mathfrak{g}_k^{\rho_{k-1}} \in H^1(S, \text{Hom}(\Omega_S, \mathcal{I}_S^k/\mathcal{I}_S^{k+1}))$  be the Čech cohomology class represented by a 1-cocycle  $\{(\mathfrak{g}_k^{\rho_{k-1}})_{\beta\alpha}\} \in H^1(\mathfrak{V}_S, \text{Hom}(\Omega_S, \mathcal{I}_S^k/\mathcal{I}_S^{k+1}))$  given by

$$(\mathfrak{g}_k^{\rho_{k-1}})_{\beta\alpha} = -\frac{1}{k!} \left. \frac{\partial^k z_\alpha^p}{\partial z_\beta^{r_1} \dots \partial z_\beta^{r_k}} \right|_S \frac{\partial}{\partial z_\alpha^p} \otimes [z_\beta^{r_1} \dots z_\beta^{r_k}]_{k+1} \in H^0(V_\alpha \cap V_\beta \cap S, \Theta_S \otimes \mathcal{I}_S^k/\mathcal{I}_S^{k+1}). \quad (61)$$

Then there exists a  $k$ -th order lifting  $\rho_k : \mathcal{O}_S \rightarrow \mathcal{O}_X/\mathcal{I}_S^{k+1}$  such that  $\rho_{k-1} = \phi_{k,k-1} \circ \rho_k$  if and only if  $\mathfrak{g}_k^{\rho_{k-1}} = 0$ . We call this  $\mathfrak{g}_k^{\rho_{k-1}}$  the obstruction to  $k$ -splitting relative to  $\rho_{k-1}$ .

**Proposition 5.2** ([1, Proposition 3.2]). Assume  $S$  is  $k$ -splitting in  $X$  and let  $\rho : \mathcal{O}_S \rightarrow \mathcal{O}_X/\mathcal{I}_S^{k+1}$  be a  $k$ -th order lifting, with  $k \geq 0$ . Then for any  $1 \leq h \leq k+1$ , the lifting  $\rho$  induces a structure of locally  $\mathcal{O}_S$ -free module on  $\mathcal{I}_S/\mathcal{I}_S^{h+1}$  for  $1 \leq h \leq k+1$  in such a way that the sequence

$$0 \longrightarrow \mathcal{I}_S^h/\mathcal{I}_S^{h+1} \longrightarrow \mathcal{I}_S/\mathcal{I}_S^{h+1} \longrightarrow \mathcal{I}_S/\mathcal{I}_S^h \longrightarrow 0 \quad (62)$$

becomes an exact sequence of locally  $\mathcal{O}_S$ -free modules.

**Definition 5.3** ([1, Definition 3.1, 3.2]). 1. If  $S$  is  $k$ -splitting in  $X$  and the sequence (62) splits for  $1 \leq h \leq k+1$ ,  $S$  is called to be  $k$ -comfortably embedded in  $X$ . Denote by  $\nu_{h-1,h} : \mathcal{I}_S/\mathcal{I}_S^h \rightarrow \mathcal{I}_S/\mathcal{I}_S^{h+1}$  the splitting  $\mathcal{O}_S$ -morphism of the sequence (62) and the comfortable splitting sequence  $\boldsymbol{\nu}_k = (\nu_{0,1}, \dots, \nu_{k,k+1})$ .

2. A  $k$ -comfortable atlas is an atlas  $\{(V_\alpha, z_\alpha)\}$  adapted to  $S$  such that

$$\frac{\partial z_\beta^p}{\partial z_\alpha^r} \in \mathcal{I}_S^k, \text{ and } \frac{\partial^2 z_\beta^r}{\partial z_\alpha^{s_1} \partial z_\alpha^{s_2}} \in \mathcal{I}_S^k \iff \frac{\partial^k z_\beta^p}{\partial z_\alpha^{r_1} \dots \partial z_\alpha^{r_k}} \Big|_S \equiv 0, \text{ and } \frac{\partial^{k+1} z_\beta^s}{\partial z_\alpha^{r_1} \dots \partial z_\alpha^{r_{k+1}}} \Big|_S \equiv 0.$$

for all  $r_1, \dots, r_k = 1, \dots, m$ , all  $p = m+1, \dots, n$ , and all indices  $\alpha, \beta$  such that  $V_\alpha \cap V_\beta \cap S \neq \emptyset$ .

**Remark 5.3.** Any submanifold  $S$  is always 0-comfortably embedded. If  $S$  is  $k$ -comfortably embedded, then  $S$  is also  $k$ -splitting.

**Theorem 5.4** ([1, Corollary 3.6]). Assume there exists a  $k$ -th order lifting  $\rho_k : \mathcal{O}_S \rightarrow \mathcal{O}_X/\mathcal{I}_S^{k+1}$  such that  $S$  is  $(k-1)$ -comfortably embedded in  $X$  with respect to  $\rho_{k-1} = \phi_{k,k-1} \circ \rho_k$ . Fix a  $(k-1)$ -comfortable pair  $(\rho_{k-1}, \boldsymbol{\nu}_{k-1})$ , and let  $\mathfrak{V} = \{(V_\alpha, z_\alpha)\}$  be a projectable atlas adapted to  $\rho_k$  and  $(\rho_{k-1}, \boldsymbol{\nu}_{k-1})$ . Then the cohomology class  $\mathfrak{h}^{\rho_k}$  associated to the exact sequence (62) is represented by 1-cocycle  $\{\mathfrak{h}_{\beta\alpha}^{\rho_k}\} \in H^1(\mathfrak{V}_S, \mathcal{N}_S \otimes \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2})$  given by

$$\mathfrak{h}_{\beta\alpha}^{\rho_k} = -\frac{1}{(k+1)!} \frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}} \dots \frac{\partial z_\beta^{s_{k+1}}}{\partial z_\alpha^{r_{k+1}}} \frac{\partial^{k+1} z_\alpha^t}{\partial z_\beta^{s_1} \dots \partial z_\beta^{s_{k+1}}} \Big|_S \partial_{z_\alpha^t} \otimes [z_\alpha^{r_1} \dots z_\alpha^{r_{k+1}}]_{k+2}.$$

**Remark 5.4.** If  $D$  is a smooth divisor, then the obstruction to  $k$ -comfortable embedding lies in  $H^1(D, N_D \otimes \mathcal{I}_D^{k+1}/\mathcal{I}_D^{k+2}) = H^1(D, (N_D)^{-k})$ . If we assume the normal bundle  $N_D$  is ample on  $D$  and  $n-1 = \dim D \geq 2$ , then the Kodaira-Nakano vanishing theorem gives  $H^1(D, (N_D)^{-k}) = 0$  for any  $k \geq 1$ . So in this case, there is no obstruction to passing from  $(k-1)$ -comfortable embedding to  $k$ -comfortable embedding (with respect to any  $k$ -splitting). Note that  $D$  is always 0-comfortably embedded. So we obtain that, if  $N_D$  is ample on  $D$  and  $\dim X \geq 3$ , then  $D$  is  $k$ -comfortably embedded, if and only if  $D$  is  $k$ -splitting, and if and only if  $D$  is  $k$ -linearizable (see Theorem 5.6).

**Theorem 5.5** ([1, Theorem 2.1, Theorem 3.5]).  $S$  is  $k$ -splitting in  $X$  if and only if there is a  $k$ -splitting atlas  $\mathfrak{V} = \{(V_\alpha, z_\alpha)\}$  of  $X$ , that is an atlas adapted to  $S$  such that

$$\begin{cases} z_\beta^r = \sum_{s=1}^m (a_{\beta\alpha})_s^r (z_\alpha)^s, & \text{for } r = 1, \dots, m, \\ z_\beta^p = \phi_{\beta\alpha}^p(z_\alpha'') + R_{k+1}^p, & \text{for } p = m+1, \dots, n, \end{cases}$$

where  $z_\alpha'' = (z_\alpha^{m+1}, \dots, z_\alpha^n)$  are local coordinates on  $S$ , and  $R_{k+1}^p$  denotes a term belong to  $\mathcal{I}_S^{k+1}$ . Furthermore,  $S$  is  $k$ -comfortably embedded in  $X$  if and only if there is a  $k$ -comfortable atlas  $\mathfrak{V} = \{(V_\alpha, z_\alpha)\}$ , that is an atlas adapted to  $S$  such that

$$\begin{cases} z_\beta^r = \sum_{s=1}^m (a_{\beta\alpha})_s^r (z_\alpha'') z_\alpha^s + R_{k+2}^r, & \text{for } r = 1, \dots, m, \\ z_\beta^p = \phi_{\beta\alpha}^p(z_\alpha'') + R_{k+1}^p, & \text{for } p = m+1, \dots, n, \end{cases}$$

where  $R_{k+2}^r \in \mathcal{I}_S^{k+2}$  and  $R_{k+1}^p \in \mathcal{I}_S^{k+1}$ .

**Theorem 5.6** ([1, Theorem 4.1]).  $S$  is  $k$ -linearizable if and only if  $S$  is  $k$ -splitting into  $X$  and  $(k-1)$ -comfortably embedded with respect to the  $(k-1)$ -th order lifting induced by the  $k$ -splitting, if and only if there is an atlas  $\mathfrak{V}$  such that the changes of coordinates are of the form:

$$\begin{cases} z_\beta^r = \sum_{s=1}^m (a_{\beta\alpha})_s^r (z_\alpha'') z_\alpha^s + R_{k+1}^r, & \text{for } r = 1, \dots, m, \\ z_\beta^p = \phi_{\beta\alpha}^p(z_\alpha'') + R_{k+1}^p, & \text{for } p = m+1, \dots, n, \end{cases}$$

where  $R_{k+1}^r, R_{k+1}^p \in \mathcal{I}_S^{k+1}$ .

### 5.3 Appendix III: Kodaira-Spencer's deformation theory

In this appendix, we recall the construction of Kodaira-Spencer class for a differentiable family using via the variation of holomorphic transition functions. Suppose we have differentiable family  $\mathcal{X} \rightarrow D$ . Suppose we have a collection of coordinate charts  $\mathfrak{U} = \{\mathcal{U}_\alpha, \{z_\alpha^i, t\}\}$  such that  $\{\mathcal{U}_\alpha\}$  is a covering of the neighborhood of  $\mathcal{X}_0$  and on each  $\mathcal{U}_\alpha$ ,  $\mathcal{X}_0 \cap \mathcal{U}_\alpha = \{t = 0\}$ . For simplicity, we can assume each  $\mathcal{U}_\alpha$  is polydisk  $D^n$ . We can write down the transition function:

$$z_\alpha^i = f_{\alpha\beta}^i(z_\beta, t), \quad t|_{\mathcal{U}_\alpha} = t|_{\mathcal{U}_\beta}.$$

The Kodaira-Spencer class is defined as follows:

$$\begin{aligned} f_{\alpha\beta}^i(f_{\beta\gamma}(z_\gamma, t), t) = f_{\alpha\gamma}^i(z_\gamma, t) &\implies \frac{\partial f_{\alpha\beta}^i(z_\beta, t)}{\partial z_\beta^j} \frac{\partial f_{\beta\gamma}^j(z_\gamma, t)}{\partial t} + \frac{\partial f_{\alpha\beta}^i(z_\beta, t)}{\partial t} \Big|_{t=0} = \frac{\partial f_{\alpha\gamma}^i(z_\gamma, t)}{\partial t} \Big|_{t=0} \\ &\implies \theta_{\beta\gamma} = \theta_{\alpha\gamma} - \theta_{\alpha\beta}, \quad \theta_{\beta\gamma} = \sum_{i=1}^n \frac{\partial f_{\beta\gamma}^i(z_\gamma, t)}{\partial t} \Big|_{t=0} \frac{\partial}{\partial z_\beta^i}. \end{aligned}$$

So we have a cocycle  $\{\theta_{\alpha\beta}\} \in \check{H}^1(\{\mathcal{U}_\alpha\}, \Theta_{\mathcal{X}_0})$  where  $\mathcal{U}_\alpha = \mathcal{U}_\alpha \cap \mathcal{X}_0$ . Now assume that  $\{\rho_\alpha\}$  is a partition of unity for the covering  $\{\mathcal{U}_\alpha\}$ . Then we can define

$$\xi_\alpha = \sum_{i=1}^n \sum_\gamma \rho_\gamma \frac{\partial f_{\alpha\gamma}^i(z_\gamma, t)}{\partial t} \Big|_{t=0} \frac{\partial}{\partial z_\alpha^i}.$$

It's easy to verify that  $\theta_{\alpha\beta} = \xi_\alpha - \xi_\beta$ , so that  $\mathbf{KS}_\mathcal{X} = \bar{\partial}\xi_\alpha = \bar{\partial}\xi_\beta$  is a globally defined  $\Theta_{\mathcal{X}_0}$ -valued closed (0,1)-form.  $\mathbf{KS}_\mathcal{X}$  represents the Kodaira-Spencer class of the deformation given by  $\mathcal{X}$ . On the other hand, we have, by the chain rule

$$\left(\frac{\partial}{\partial t}\right)_\beta = \sum_{i=1}^n \frac{\partial f_{\alpha\beta}^i(z_\beta, t)}{\partial t} \frac{\partial}{\partial z_\alpha^i} + \left(\frac{\partial}{\partial t}\right)_\alpha, \quad \frac{\partial}{\partial z_\beta^j} = \sum_{i=1}^n \frac{\partial f_{\alpha\beta}^i(z_\beta, t)}{\partial z_\beta^j} \frac{\partial}{\partial z_\alpha^i}.$$

We define the differentiable vector field locally by:

$$\begin{aligned} \mathbb{V} &= \sum_\gamma \rho_\beta \left(\frac{\partial}{\partial t}\right)_\beta = \sum_\beta \rho_\beta \sum_{i=1}^n \frac{\partial f_{\alpha\beta}^i(z_\beta, t)}{\partial t} \frac{\partial}{\partial z_\alpha^i} + \left(\frac{\partial}{\partial t}\right)_\alpha \\ &= \sum_{i=1}^n \left( \sum_\beta \rho_\beta \frac{\partial f_{\alpha\beta}^i(z_\beta, t)}{\partial t} \right) \frac{\partial}{\partial z_\alpha^i} + \left(\frac{\partial}{\partial t}\right)_\alpha. \end{aligned}$$

Then  $\mathbb{V}$  is a globally defined vector field in the neighborhood of  $\mathcal{X}_0$ . Let  $\sigma(t)$  be the flow associated with  $\mathbb{V}$  which exists for sufficiently small  $t$ . We have the identity:

$$\frac{d}{dt}(\sigma(t)^* J) = (\mathcal{L}_\mathbb{V} J)(\partial_{\bar{z}^j}) d\bar{z}^j = \bar{\partial}\mathbb{V}.$$

Notice that  $\bar{\partial}\mathbb{V}|_{t=0} = \bar{\partial}\xi_\alpha = \mathbf{KS}_\mathcal{X}$ .

### 5.4 Appendix IV: Deformation of complex cones

Here we recall Schlessinger's work in [27], [28] on the deformation of normal isolated singularities. Assume  $D$  is a projective Kähler manifold with a positive line bundle  $L \rightarrow D$ . Consider the affine cone:

$$C := C(D, L) = \text{Spec} \bigoplus_{k=0}^{+\infty} H^0(D, L^{\otimes k}). \quad (63)$$

$C$  has a normal isolated singularity at the vertex  $\mathfrak{o}$ .

**Proposition 5.3** ([27], [28]). *Assume  $C$  is embedded into  $\mathbb{C}^N$ . We have an exact sequence:*

$$H^0(U, \Theta_{\mathbb{C}^N}|_U) \rightarrow H^0(U, N_U) \rightarrow \mathbf{T}_C^1 \rightarrow 0 \quad (64)$$

$$0 \rightarrow \mathbf{T}_C^1 \rightarrow H^1(U, \Theta_U) \rightarrow H^1(U, \Theta_{\mathbb{C}^N}|_U) \quad (65)$$

*Proof.* We have the conormal exact sequence:

$$\mathcal{I}_C/\mathcal{I}_C^2 \rightarrow \Omega_{\mathbb{C}^N}|_C \rightarrow \Omega_C \rightarrow 0,$$

whose dual defines the sheaf  $\mathcal{T}^1$ :

$$0 \rightarrow \Theta_C \rightarrow \Theta_{\mathbb{C}^N}|_C \rightarrow N_C \rightarrow \mathcal{T}^1 \rightarrow 0.$$

In particular, the first three sheaves are reflexive. Because  $C$  is affine, by definition we get the exact sequence:

$$0 \rightarrow H^0(C, \Theta_C) \rightarrow H^0(C, \Theta_{\mathbb{C}^N}|_C) \rightarrow H^0(C, N_C) \rightarrow \mathbf{T}_C^1 \rightarrow 0. \quad (66)$$

Because  $C$  is normal, by Serre's criterion for normality,  $C_D$  has depth  $\text{depth}_{\mathfrak{c}} C \geq 2$  at its vertex. Because the first three sheaves are reflexive, by [28, Lemma 1], the depth of each is  $\geq 2$ . So in (66) we can replace  $H^0(X, \cdot)$  by  $H^0(U, \cdot)$  to get:

$$0 \rightarrow H^0(U, \Theta_U) \rightarrow H^0(U, \Theta_{\mathbb{C}^N}|_U) \rightarrow H^0(U, N_U) \rightarrow \mathbf{T}_C^1 \rightarrow 0, \quad (67)$$

On the other hand, we have

$$0 \rightarrow \Theta_U \rightarrow \Theta_{\mathbb{C}^N}|_U \rightarrow N_U \rightarrow 0,$$

which gives us the exact sequence:

$$0 \rightarrow H^0(U, \Theta_U) \rightarrow H^0(U, \Theta_{\mathbb{C}^N}|_U) \rightarrow H^0(U, N_U) \rightarrow H^1(U, \Theta_U) \rightarrow H^1(U, \Theta_{\mathbb{C}^N}|_U). \quad (68)$$

Combining (67) and (68), we get (64) and (65).  $\square$

As an example of the above general theory, consider a projective manifold  $D \subset \mathbb{P}^{N-1}$ . We assume that  $D$  is projectively normal in  $\mathbb{P}^{N-1}$  so that the affine cone over  $D$  is normal and is equal to  $C_D = C(D, H)$  where  $H$  is the hyperplane bundle of  $\mathbb{P}^{N-1}$ . Then it's easy to verify that (see [27], [4]):

$$H^0(U, \Theta_{\mathbb{C}^N}|_U) = \sum_{j=-\infty}^{+\infty} H^0(D, \mathcal{O}_D(j+1)), \quad H^0(U, N_U) = \sum_{j=-\infty}^{+\infty} H^0(D, N_D(j)).$$

Decompose  $\mathbf{T}_C^1 = \sum_{j=-\infty}^{+\infty} \mathbf{T}_C^1(j)$  into weight spaces. Then by (64) we have the exact sequence:

$$H^0(D, \mathcal{O}_D(j+1))^N \xrightarrow{\text{Jac}} H^0(D, N_D(j)) \longrightarrow \mathbf{T}_C^1(j) \rightarrow 0. \quad (69)$$

**Example 5.1** (cf. [4, Section 4]). Assume  $D^{n-1} \subset \mathbb{P}^{N-1}$  is a complete intersection

$$D = \bigcap_{i=1}^{N-n} \{F_i = 0\} \subset \mathbb{P}^{N-1},$$

where  $F_i$  is a homogeneous polynomial of degree  $d_i$ . We assume  $\{Z_1, \dots, Z_N\}$  are homogeneous coordinates of  $\mathbb{P}^{N-1}$  and denote

$$R(D, H) = \bigoplus_{m=0}^{+\infty} H^0(D, mH) \cong \mathbb{C}[Z_1, \dots, Z_N]/\langle F_1, \dots, F_{N-n} \rangle.$$

Note that this is nothing but the affine coordinate ring of  $C(D, H)$ . Then

$$H^0(D, \mathcal{O}_D(j+1)) = H^0(D, (j+1)H) = R(D, H)(j+1);$$

$$H^0(D, N_D(j)) = \bigoplus_{i=1}^{N-n} H^0(D, (d_i + j)H) = \bigoplus_{i=1}^{N-n} R(D, H)(d_i + j).$$

The map

$$\text{Jac} : R(D, H)(j+1)^N \rightarrow \bigoplus_{i=1}^{N-n} R(D, H)(d_i + j)$$

is given by the Jacobian matrix  $(\partial F_k / \partial Z^l)_{k=1, \dots, N-n}^{l=1, \dots, N}$ , with the quotient:

$$\mathbf{T}_C^1(j) = \frac{\bigoplus_{i=1}^{N-n} R(D, H)(d_i + j)}{\text{Jac}(R(D, H)(j+1)^{\oplus N})}.$$

Now assume  $\mathcal{G} = \{g_i = g_i(z_1, \dots, z_N), i = 1, \dots, N-n\}$  consists of (not necessarily homogeneous) polynomials. We can consider the deformation of  $C(D, H) \subset \mathbb{C}^N$  given by:

$$\mathcal{C}_t = \bigcap_{i=1}^{N-n} \{F_i(z_1, \dots, z_N) + tg_i = 0\} \subset \mathbb{C}^N.$$

If we assume image  $[\mathcal{G}]$  in  $\mathbf{T}_C^1$  is not zero, then by (69), we see that the weight of this deformation is the weight of  $[\mathcal{G}]$ . Note that the polynomials in the image of Jac have degree  $\geq d_i - 1$ . So if  $g_i$  is of degree  $e_i \leq d_i - 2$ , it's easy to see that the  $[\mathcal{G}]$  is indeed not zero and the weight is equal to  $\max\{e_i - d_i\} = -\min\{d_i - e_i\}$ .

**Remark 5.5.** The reason that we assume the non vanishing of  $[\mathcal{G}]$  is to guarantee the induced map  $\mathbb{C} \rightarrow \mathbf{T}_C^1$  does not have a vanishing 1st order derivative. Otherwise, we need to consider higher derivatives (to obtain reduced Kodaira-Spencer class) as the following example shows:

$$\{z_1^2 + z_2^2 + z_3^2 = 0\} \rightsquigarrow \{z_1^2 + z_2^2 + z_3^2 + tz_3 = 0\}.$$

We have  $\mathbf{T}_C^1 = \mathbb{C}[z_1, z_2, z_3] / \langle z_1, z_2, z_3 \rangle$ . So  $\mathcal{G} = (g = z_3)$  gives vanishing image  $[\mathcal{G}] = 0$ . However, we have:

$$\{z_1^2 + z_2^2 + z_3^2 + tz_3 = 0\} = \{z_1^2 + z_2^2 + (z_3 + t/2)^2 - \frac{t^2}{4} = 0\} \cong \{z_1^2 + z_2^2 + \tilde{z}_3^2 - \frac{t^2}{4} = 0\}.$$

So the induced map  $\mathbb{C} \rightarrow \mathbf{T}_C^1$  vanishes to the 2nd order at 0 and the weight of the deformation is actually equal to  $-2$ .

Finally we briefly recall Pinkham's results on deformation of isolated singularities with  $\mathbb{C}^*$  actions. We state the result in our setting of affine cones.

**Theorem 5.7** ([25, 26]). 1. There exists a formal versal  $\mathbb{C}^*$  equivariant deformation  $\mathcal{C} \rightarrow V$  of  $C$ .

2. Let  $\mathcal{Y} \rightarrow T$  be any formal  $\mathbb{C}^*$  equivariant deformation of  $X$ . Then there exists a  $\mathbb{C}^*$  equivariant morphism  $\phi : T \rightarrow V$  and a  $\mathbb{C}^*$  equivariant isomorphism of the deformation  $\mathcal{Y} \rightarrow T$  with the pull back  $\mathcal{X} \times_V T \rightarrow T$ .

Let  $t_j$  be homogeneous generators of the maximal ideal of weight  $d(t_j)$ . Let  $J^-$  be the ideal in  $\mathcal{O}_V$  generated by  $\{t_j; d(t_j) < 0\}$ . Let  $V^-$  be the subvariety defined by  $J^-$ .

**Theorem 5.8** ([26, Theorem 2.9]).  $\mathcal{C}^- \rightarrow V^-$  extends to a proper flat family  $\overline{\mathcal{C}}^- \rightarrow V^-$  of deformations of  $\overline{C}$ .  $\overline{\mathcal{C}}^- - \mathcal{C}^- \cong D_\infty \times V^-$  and  $\overline{\mathcal{C}}^- \rightarrow V^-$  is a locally trivial deformation near  $D_\infty$ .

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Mathematics Department, Stony Brook University, Stony Brook NY, 11794-3651, USA  
Email: chi.li@stonybrook.edu